Positronium collapse in hypercritical magnetic field and restructuring of the vacuum in QED

A E Shabad* and V V Usov†

* P.N.Lebedev Physics Institute, Moscow, Russia

† Weizmann Institute of Science, Rehovot, Israel

E-mail: * `shabad@lpi.ru`,
† `fnusov@wicc.weizmann.ac.il`

Abstract. The Bethe-Salpeter equation in a strong magnetic field is studied for positronium atom in an ultra-relativistic regime, and a (hypercritical) value for the magnetic field is determined, which provides the full compensation of the positronium rest mass by the binding energy in the maximum symmetry state. The compensation becomes possible owing to the falling to the center phenomenon. The relativistic form in two-dimensional Minkowsky space is derived for the four-dimensional Bethe-Salpeter equation in the limit of an infinitely strong magnetic field, and used for finding the above hypercritical value. Once the positronium rest mass is compensated by the mass defect the energy barrier separating the electron-positron system from the vacuum disappears. We thus describe the structure of the vacuum in terms of strongly localized states of tightly mutually bound (or confined) pairs. Their delocalization for still higher magnetic field, capable of screening its further growth, is discussed.

1. Introduction

In the present paper we are considering, in the framework of quantum electrodynamics, the phenomenon of falling to the center in a system of two charged particles caused by the ultraviolet singularity $1/x^2$ of the photon propagator - which mediates the interaction between them - on the light-cone, $x^2 = x_0^2 - \mathbf{x}^2 \simeq 0$. Here x_0 is the time, and $\mathbf{x}$ is the space coordinate. This phenomenon occurs in a number of problems. In some of cases listed below the matter comes to the one-dimensional Schrödinger equation with the singular attractive potential $U(r) = -\beta/r^2$, $0 < r < \infty$, the falling to the center depending on the value of the constant β that fixes the strength of the coupling [1].

Long ago it was established that the falling to the center is inherent in the Bethe-Salpeter equation for electron-positron system treated in a fully relativistic way and taken in the ladder approximation, provided that the fine structure constant α exceeds a critical value $\alpha_{\rm cr}$, $(1/137) \ll \alpha_{\rm cr} < 1$. We refer to the so-called Goldstein solution [2] of the Bessel differential equation in the variable $s = \sqrt{-x^2}$, the space-like interval between the particles, to which the problem is reduced in the ultra-relativistic limit. Falling to the center implies that the singular attraction is so strong that the energy

spectrum becomes unlimited from below, and hence - once the singular problem is duely defined - the binding energy may completely cancel the rest mass, so that the gap between positive and negative continua disappears. This reminds to some extent the situation in the problem of an electron in an external electric field, occupying the whole space, where the tunnelling through the gap leads to the vacuum instability with respect to e^+e^-−pair creation known as Schwinger effect, although the latter does not have a threshold character. The value $\alpha = \alpha_{\rm cr}$ may delimit the range of self-consistency of the theory known as quantum electrodynamics, at least in its customary form.

Another well-known theater [3], [4], [5] where the falling to the center acts is the point-like nucleus with its charge Z exceeding the critical value $Z_{\rm cr} = \alpha^{-1} = 137$, into whose Coulomb field a relativistic electron described by the Dirac equation is placed. After this problem is properly regularized, the electron energy level reaches the border $\varepsilon = -m$ of the lower continuum as the charge Z grows, which gives rise to electron-positron pair production, the free positrons leaving the atom. This mechanism leads to diminishing the charge and restores it at the critical value that cannot thus be exceeded.

If an electron in an atom is treated within the Schrödinger equation, or a pair of particles is addressed in a semi-relativistic way with the aid of the Bethe-Salpeter equation, wherein the famous Salpeter's equal-time Ansatz is made (resulting in the disregard of the retardation effects), the falling to the center does not take place. The matter is that the singularity in the point $r = 0$, $r = \sqrt{\mathbf{x}^2}$ of the Coulomb potential $U(r) = -\alpha Z/r$ originating in this case from the light-cone singularity of the photon propagator, mentioned above, is not sufficiently strong, and the energy level remains shallow. On the contrary, in the above two cases the falling to the center may be attributed to relativistic enhancement of the Coulomb force - the second-order differential equation to which the spherically symmetric Dirac equation may be reduced by excluding the second spinor component, contains the sufficiently singular attractive term $-(\alpha Z/r)^2$ in the effective potential [3].

The situation changes drastically when a strong external magnetic field $\mathbf{B}$ is applied to the system. Already the Schrödinger equation for a particle situated in the combination of the Coulomb and strong magnetic fields possesses the falling to the center caused by the singularity $1/|z|$ in the differential Schrödinger equation defined on the whole axis $-\infty < z < \infty$. Here z is the electron coordinate along the axis parallel to the magnetic field. The energy spectrum of this equation contains large negative values tending to minus infinity [6] as the magnetic field $B \to \infty$. The reason lies in the dimensional reduction: for strong magnetic field the electron is restricted to the lowest Landau level, consequently its wave function obeys a two-dimensional (one space and one time coordinates) differential equation. The reduction of the number of degrees of freedom causes the effective strengthening of the attraction, like in the quark confinement problem. Analogous strengthening takes place when the electron-positron system is described by the Bethe-Salpeter equation studied using the semi-relativistic Ansatz that implies the non-relativistic character of the relative motion of the constituent particles [7] - [11]. Again, the energy spectrum of the corresponding two-

dimensional equation is unlimited from below as the magnetic field grows. The total rest mass of the ground state is compensated when the magnetic field reaches the value [9], whose order of magnitude is determined by the large factor $\exp(\mathrm{const}/\alpha)$. As pointed in [9] those results, however, are not reliable, since they depend on the unrighteous extrapolation of non-relativistic procedure to the essentially relativistic region of large mass defect.

The relativistic treatment of an electron in a combination of external Coulomb and magnetic field was given in [12]. The result was that if the magnetic field is reasonably strong ($\sim 10^{15}$ G) the critical value of the nucleus charge Z may correspond to stable elements with $Z \sim 90$.

In the present paper we consider the system of two charged relativistic particles - especially the electron and positron - in interaction with each other, when placed in a strong constant and homogeneous magnetic field B. To this end we use the Bethe-Salpeter equation in the ladder approximation without exploiting any non-relativistic assumption. We derive its limiting - when $B \to \infty$ - form, which comes out to be a Bethe-Salpeter equation in two-dimensional Minkowsky space-time, covariant under the corresponding Lorentz subgroup - the boost along the magnetic field. Stress, that the two-dimensionality holds only with respect to the degrees of freedom of charged particles, while the photons remain 4-dimensional in the sense that the singularity of the photon propagator is determined by the inverse d'Alambert operator in the 4-dimensional, and not two-dimensional Minkowsky space. (Otherwise it would be weaker). The term responsible for interaction with an arbitrary electric field directed along $\mathbf{B}$ is also included and does not lay obstacles to the dimensional reduction. Throughout the paper we set $\hbar = c = 1$ and use the Heaviside-Lorentz units, where $\alpha = e^2/4\pi$. $B|_{\mathrm{Gaussian}} = \sqrt{4\pi}\, B|_{\mathrm{Heaviside-Lorentz}}$.

We make sure that in the case under consideration the critical value of the coupling constant is zero, $\alpha_{\mathrm{cr}} = 0$, i.e., the falling to the center is present already for its genuine value $\alpha = 1/137$, in contrast to the no-magnetic-field case, where $\alpha_{\mathrm{cr}} > 1/137$. If the magnetic field is large, but finite, the dimensional reduction holds everywhere except a small neighborhood of the singular point $s = 0$, wherein the mutual interaction between the particles dominates over their interaction with the magnetic field. The dimensionality of the space-time in this neighborhood remains to be 4, and its size is determined by the Larmour radius $L_B = (eB)^{-1/2}$ that is zero in the limit $B = \infty$. The latter supplies the singular problem with a regularizing length. The larger the magnetic field, the smaller the regularizing length, the deeper the level. We find the value of the magnetic field - we call it *first hypercritical field* -

$$B^{(1)}_{\mathrm{hpcr}} = \frac{m^2}{4e} \exp\left\{\frac{\pi^{3/2}}{\sqrt{\alpha}} + 2C_{\mathrm{E}}\right\}, \tag{1}$$

where $C_{\mathrm{E}} = 0.577$ is the Euler constant, that provides disappearance of the center-of-mass energy of the electron-positron pair and of its center-of-mass momentum component along $\mathbf{B}$. We refer to this situation as a collapse of positronium.

In discussing the physical consequences of the falling to the center we appeal to the

approach recently developed by one of the present authors as applied to the Schrödinger equation with singular potential [13] and to the Dirac equation in supercritical Coulomb field [5]. Within this approach the singular center looks like a black hole. The solutions of the differential equation that oscillate near the singularity point are treated as free particles emitted and absorbed by the singularity. This treatment becomes natural after the differential equation is written as the generalized eigenvalue problem with respect to the coupling constant. Its solutions make a Hilbert space and are subject to orthonormality relations with a singular measure. This singularity makes it possible for the oscillating solutions to be normalized to δ-functions, as free particle wave-functions should be. The nontrivial, singular measure that appears in the definition of the scalar product of quantum states in the Hilbert space of quantum mechanics introduces the geometry of a black hole of non-gravitational origin and the idea of horizon. The deviation from the standard quantum theory manifests itself in this approach only when particles are so close to one another that the mutual Coulomb field they are subjected to falls beyond the range, where the standard theory may be referred to as firmly established [5].

Within this approach the regularizing length provided by the Larmour radius is dealt with not as a cut-off, but as a lower border of the normalization volume, the event horizon in a way. Although the result (1) is obtained following the concept of Refs. [13], [5], it can be reproduced without essential alteration within the standard cut-off philosophy, too.

The most intriguing question is what happens after the magnetic field exceeds the first hypercritical value (1). The solution of the Bethe-Salpeter equation in two-dimensional space-time in the ultra-relativistic limit studied in the present paper corresponds to formation of special "confined" states in the kinematical domain called sector III in [13], [5]. (Within the standard approach these would be bound states, although this is less adequate). As the corresponding overall energy and momentum of such e^+e^--state is zero, it is not separated from the vacuum by an energy barrier. Besides, this state is the one of maximum symmetry in the coordinate and spin space. Hence, it may be thought of as relating to the vacuum, as well, and describing its structure. The confined particles cannot escape to infinite distance from one another, on the contrary the probability density of the confined state is concentrated near the point $s = 0$, behind the horizon - as distinct from the ordinary bound state.

The situation is expected to change as the magnetic field goes on growing. At a certain stage - we reserve the name *second hypercritical* for the corresponding value of the magnetic field - deconfinement of the above strongly localized states may occur. The corresponding solutions to the Bethe-Salpeter equation are not yet strictly obtained, which makes us describe the deconfinement more hypothetically. After the level deepens further, the center-of-mass 2-momentum gets into sufficiently far space-like region, and solutions oscillating at large distances between electron and positron appear. Thus, the state delocalizes. The delocalized electron-positron pairs produced from the vacuum, each particle on a Larmour orbit, should screen the magnetic field and stop its growing

above the second hypercritical value, this value being the absolute maximum of the magnetic field admitted within quantum electrodynamics. Simultaneously, the space-like total momentum provides the lattice structure to the vacuum.

This resembles the case of supercritical nucleus where there are states (that belong to sector IV in the nomenclature of Refs.[13], [5]) admitting the leakage to infinity, which provides the mechanism for reducing the charge of the nucleus below the critical value.

No sooner than the delocalized states are found in our present problem one may definitely claim the instability of the vacuum with the second hypercritical magnetic field or - which is the same - the instability of such field under the pair creation that might provide the mechanism for its diminishing. For the present, we state that the first hypervalue (1) is such a value of the magnetic field, the exceeding of which would already cause restructuring of the vacuum and demand a profound revision of quantum electrodynamics.

The paper is organized as follows. In Section 2 we revisit Goldstein's solution by referring to various possibilities in approaching the falling to the center, especially the one invoked by the previous work [13], [5].

In Section 3 we derive the ultimate two-dimensional form of the Bethe-Salpeter equation in its differential version characteristic of the ladder approximation, when the magnetic field tends to infinity, with the help of expansion over the complete set of Ritus matrix eigenfunctions [14]. The latter accumulate the spacial and spinor dependence on the transversal-to-the-field degree of freedom. The Fourier-Ritus transform of the Bethe-Salpeter amplitude obeys an infinite chain of coupled differential equations that decouple in the limit of large B, so that we are left with one closed equation for the amplitude component with the Landau quantum numbers of the electron and positron both equal to zero, while the components with other values of Landau quantum numbers vanish in this limit. The resulting equation is a differential equation with respect to two variables that are the differences of the particle coordinates: along the time $t = x_0^{\rm e} - x_0^{\rm p}$ and along the magnetic field $z = x_3^{\rm e} - x_3^{\rm p}$. It contains only two Dirac matrices γ_0 and γ_3 and can be alternatively written using 2×2 Pauli matrices. Arbitrary external electric field $\mathbf{E}$ along $\mathbf{B}$ is also included, $E \ll B$. By introducing different masses the resulting two-dimensional equation is easily modified to cover also the case of an one-electron atom in strong magnetic field and/or other pairs of charged particles.

In Section 4 the ultra-relativistic solutions (possessing maximum symmetry) to the equation derived in Section 3 are depicted corresponding to the vanishing energy-momentum of the e^+e^--state, and the first hypercritical magnetic field is found basing on the standing wave boundary condition imposed on the lower border of the normalizing volume - as prescribed by the theory in Refs. [13], [5]. Also the standard cut-off procedure of Refs. [1], [3] is fulfilled to give practically the same value (1). Further, we estimate possible modifications that might be introduced by radiative corrections to the mass operator, to find that these cannot change the conclusions any essentially, and discuss the deconfinement.

2. Bethe-Salpeter equation for positronium in ultrarelativistic regime

The fully relativistic Bethe-Salpeter equation for a system of two Fermions of masses m_{a} and m_{b} and opposite charges has in the ladder approximation the form [15]

$$[(\mathrm{i}\widehat{\partial}_{\mathrm{a}} - m_{\mathrm{a}})]_{\alpha\beta}[(\mathrm{i}\widehat{\partial}_{\mathrm{b}} - m_{\mathrm{b}})]_{\mu\nu}\ \chi_{\ \beta\nu}(x_{\mathrm{a}}, x_{\mathrm{b}}) \ = -\mathrm{i}8\pi\alpha D_{mn}(x_{\mathrm{a}} - x_{\mathrm{b}})\gamma^m_{\alpha\beta}\gamma^n_{\mu\nu}\ \chi_{\beta\nu}(x_{\mathrm{a}}, x_{\mathrm{b}}) \ . \quad (2)$$

Here γ's are the Dirac gamma-matrices, the two-time wave function $\chi(x_{\mathrm{a}}, x_{\mathrm{b}})$ is a 4×4 matrix with respect to spinor indices, the corresponding Greek letters running the values (1,2,3,4). The summation over the repeated vector indices $m, n = 0, 1, 2, 3$ is also meant. The derivatives

$$\widehat{\partial}_{\mathrm{a,b}} = \gamma_0\partial^0_{\mathrm{a,b}} - \gamma_i\partial^i_{\mathrm{a,b}}, \quad i = 1, 2, 3 \quad (3)$$

act on the first and the second arguments of $\chi(x_{\mathrm{a}}, x_{\mathrm{b}})$, respectively, α is the fine structure constant, and $D_{mn}(x_{\mathrm{a}} - x_{\mathrm{b}})$ is the photon propagator. Note that the Feynman photon Green function D_{F} of Ref.[15] is defined as $2D$. The translational invariance implies that the solution, which is an eigenfunction of the translation operator, should have the form

$$\chi_P(x_{\mathrm{a}}, x_{\mathrm{b}}) \ = \exp\left(\mathrm{i}P\frac{x_{\mathrm{a}} + x_{\mathrm{b}}}{2}\right)\ \eta_P(x), \quad (4)$$

where $x = x_{\mathrm{a}} - x_{\mathrm{b}}$.

Equation (2) is a complicated set of differential equations, which can, however, be essentially simplified, if one assumes $P^\mu = 0$ as explained in the previous subsection. In this case, equation (2) can be transcribed as

$$(\mathrm{i}\overrightarrow{\widehat{\partial}} - m_{\mathrm{a}})\eta_0(x)(-\mathrm{i}\overleftarrow{\widehat{\partial}}^{\mathrm{T}} - m_{\mathrm{b}}) = -\mathrm{i}8\pi\alpha D_{mn}(x)\gamma^m\eta_0(x)(\gamma^n)^{\mathrm{T}}, \quad (5)$$

where the superscript T means transposition and the derivative acts on the relative variable x to the right or to the left of it according to what is prescribed by the direction of the arrow. With the help of the known relation [4]

$$\gamma_n^{\mathrm{T}} = -C^{-1}\gamma_n C, \quad (6)$$

where C is the charge conjugation matrix, eq.(5) is transformed to

$$(\mathrm{i}\overrightarrow{\widehat{\partial}} - m_{\mathrm{a}})\eta_0(x)C^{-1}(\mathrm{i}\overleftarrow{\widehat{\partial}} - m_{\mathrm{b}}) = \mathrm{i}8\pi\alpha D_{mn}(x)\gamma^m\eta_0(x)C^{-1}\gamma^n. \quad (7)$$

The Bethe-Salpeter amplitude $\eta_0(x)C^{-1}$ is the one for the case where the two Fermions are Dirac conjugated to one another, i.e. are electron and positron, equation (7) with $m_{\mathrm{a}} = m_{\mathrm{b}} = m$ being just the Bethe-Salpeter equation for electron-positron system [4].

We take the photon propagator in the Feynman gauge $D_{mn}(x) = g_{mn}D(x^2)$, where the metric tensor obeys diag $g_{mn} = (1, -1, -1, -1)$. The Ansatz

$$\eta_0(x) = \Theta(x)\gamma_5 C, \quad (8)$$

where $\Theta(x)$ is a unit matrix containing a single function of x, is then consistent with the set (7) and turns it into the equation

$$(-\Box + m^2)\Theta(x) + 32\mathrm{i}\alpha\pi D(x^2)\Theta(x) = 0, \quad (9)$$

where $\Box = -\partial_0^2 + \Delta$ is the Laplace operator. The figure 32 here has resulted from the multiplication of 8 in (7) by 4, which is associated with the dimension of the space:

$$\sum_{m,n=0,1,2,3} g_{mn}\gamma_m\gamma_n = 4 \tag{10}$$

The photon propagator is singular on the light cone $x^2 = 0$:

$$D(x^2) = \frac{-\mathrm{i}}{4\pi^2 x^2}. \tag{11}$$

In the space-like region $x^2 < 0$, for the most symmetrical state where the solution does not depend on the angles in the 4-dimensional Euclidean space, $\Theta(x) = \Theta(s)$, $s = \sqrt{-x^2}$, equation (9) with eq.(11) for $D(x^2)$ becomes the Bessel differential equation

$$-\frac{\mathrm{d}^2\Theta}{\mathrm{d}s^2} - \frac{3}{s}\frac{\mathrm{d}\Theta}{\mathrm{d}s} + m^2\Theta = \frac{8\alpha}{\pi s^2}\Theta. \tag{12}$$

Its solution is known as Goldstein's solution [2] to the Bethe-Salpeter equation with $P_\mu = 0$. (See the review [16], where other ultrarelativistic solutions, corresponding to Ansätze different from the above are also listed). Near the singular point $s = 0$ equation (12) has the asymptotic form

$$-\frac{\mathrm{d}^2\Theta}{\mathrm{d}s^2} - \frac{3}{s}\frac{\mathrm{d}\Theta}{\mathrm{d}s} = \frac{8\alpha}{\pi s^2}\Theta(s), \tag{13}$$

which is also the asymptotic form of the full (with $P_\mu \neq 0$) Bethe-Salpeter equation (2). Its solutions behave near the singular point $s = 0$ like s^σ, where

$$\sigma = -1 \pm \sqrt{1 - \frac{8\alpha}{\pi}} \tag{14}$$

The substitution $\Theta(s) = \Psi(s)s^{-\frac{3}{2}}$ eliminates the first derivative and reduces equation (12) to the standard form,

$$-\frac{\mathrm{d}^2\Psi}{\mathrm{d}s^2} - \frac{1}{s^2}\left(\frac{8\alpha}{\pi} - \frac{3}{4}\right)\Psi = -m^2\Psi, \quad 0 < s < \infty, \tag{15}$$

of a Schrödinger-like equation with purely centrifugal - attractive or repulsive, depending on the sign of the difference $(\frac{8\alpha}{\pi} - \frac{3}{4})$ - potential. Its solutions behave near the singular point $s = 0$ like

$$\Psi(s) \ \sim \ s^{\frac{1}{2} \pm \sqrt{1 - \frac{8\alpha}{\pi}}}. \tag{16}$$

The same as for the usual radial Schrödinger equation, the natural mathematical requirement that the norm $\int_0 |\Psi(s)|^2 \mathrm{d}s$ be convergent at the lower limit $s = 0$ is not yet sufficient to rule out the more singular solution, which corresponds to the lower sign in (16), in the whole range where the square root in (14), (16) is real (in the present case this requirement would separate the less singular solution only for formally negative α !). To do so an additional physical requirement concerning the behavior of the wave function near the origin is usually imposed to fix the eigenvalue problem in quantum mechanics [1]. For the Bethe-Salpeter equation such physical requirement was

established by Mandelstam [17]. It reads that the integral over a small three-dimensional closed hypersurface $S(3)$ of the 4-vector current density $\overline{\Theta}(x)\partial_\mu\Theta(x)$ around the origin

$$\oint \overline{\Theta}(x)\partial_\mu\Theta(x)\mathrm{d}\sigma_\mu \simeq s^{\pm 2\sqrt{1-\frac{8\alpha}{\pi}}} \int \mathrm{d}\Omega \tag{17}$$

should tend to zero together with the radius s of the hypersphere $S(3)$ in the Euclidean 4-space-time. This implies that $\Theta(s)$ should increase slower than s^{-1} and makes only solutions with the upper sign acceptable, provided that $\alpha < \alpha_{\rm cr}$, where

$$\alpha_{\rm cr} = \frac{\pi}{8}. \tag{18}$$

The above requirement also rules out the both oscillating solutions when $\alpha \geq \alpha_{\rm cr}$. If, however, one keeps to a weaker condition that the current flow (17) be *finite* as $s \to 0$ and, correspondingly, $\Theta(s)$ increase no faster than s^{-1}, the both oscillating solutions are compatible with it (note the complex conjugation sign in (17), due to which the square root does not appear in it if $\alpha \geq \alpha_{\rm cr}$). Such situation is typical of the falling to the center phenomenon.

The full Bethe-Salpeter equation (2) certainly has bound states, when $\alpha < \alpha_{\rm cr}$, corresponding to positronium atom, and the above condition serves to define them. The binding energy of the realistic, $\alpha = 1/137$, positronium makes about a millionth fraction of its rest mass. On the contrary, there is no bound state described by equation (12). The exact solution to the latter, decreasing at infinity, is expressed in terms of the McDonald function as $\Theta(s) = (1/s)K_{\sqrt{1-\frac{8\alpha}{\pi}}}(ms)$. Its asymptotic behavior near $s = 0$ is a linear combination of the both terms (14) and, hence this solution is forbidden by the boundary conditions discussed above, provided that $\alpha < \alpha_{\rm cr}$. This means that no bound state exists with $P_0 = \mathbf{P} = 0$ for the coupling, smaller than the critical value $\alpha = \alpha_{\rm cr}$ (18), and the gap between electrons and positrons survives down to this value.

Our main concern is about what happens in the overcritical region of the coupling constant $\alpha \geq \alpha_{\rm cr}$. Three different approaches have the right to exist for treating this case.

The first is the same as the one used for considering a Dirac electron in the Coulomb field of a nucleus with its charge greater than $1/\alpha$, $Z > 137$ (see [3], [4]). In that approach the finite size of the nucleus was exploited as providing a natural cut-off to the singular potential. There is no such natural fundamental length in our problem, but if we introduce it formally, for instance by shifting the pole in the photon propagator (11) from the light cone inwards the time-like domain, $D_{mn}(x^2) \asymp 1/(x^2 - \lambda^2)$ we would come to the situation where the positronium gradually approaches the point $P_0 = \mathbf{P} = 0$ as α grows and reaches it at certain $\alpha_{\rm cr} = \alpha(\lambda)$. In other words, where there is falling to the center, the attraction is so strong that for a sufficient value of the coupling the level becomes so deep as to fully compensate for the whole mass of the positronium atom. When analogous situation occurs in the above case of the supercharged nucleus, the electron level dips into the Dirac sea of positron states, the vacuum becomes unstable with respect to e^+e^--pair creation. The essential disadvantage of this approach is that all important quantities, the pair production probability among them, depend on the

cut-off length and do not have a definite limit when $\lambda \to 0$. This fact makes the results doubtful after the cut-off length becomes less than the characteristic length of the problem, which is the electron Compton length m^{-1}. This is the case for the (supercritical) nucleus, whose size is adopted to be $10^{-12}cm \ll m^{-1}$. Down to what other border should one believe the result once it does not converge?

The second approach might be dependent on the von Neuman technique of the so called self-adjoint extension of the Hamiltonian (see the pioneering works [18], [19] and the monograph [20]). According to [18] and [19] there is an infinite number of discrete eigenvalues of the operator, whose differential expression is determined by the left-hand side of equation (15). These extend to $m^2 = \infty$ for fixed α. The self-adjoint extension is defined up to an arbitrary parameter that can always be chosen in such a way as to make any given $-m^2$ - the electron mass squared - an eigenvalue. Then one should conclude that the tightly bound states exist beyond the point $\alpha = \pi/8$. The disadvantage of this approach is in that no physical criterion is known to fix the arbitrariness of the self-adjoint extension [20]. If one likes to take the above prescription seriously, one would have to alter the choice of this parameter when going to a different α or m^2. We do not know if the method of self-adjoint extension was ever applied to the problem under consideration. In the problem of supercritical nucleus the application of this method yields the result that the electron level never sinks into the continuum [18], hence there is no place for the electron-positron pairs production effect.

A third approach is, in our opinion, most adequate. It is to treat equation (15) as an eigenvalue problem with respect to the coupling constant α. By putting this eigenvalue problem in the form of an integral equation it was demonstrated [21], [16] that the corresponding integral kernel gives rise to a self-adjoint operator, once an appropriate norm is finite, and the eigenvalues $\alpha < \alpha_{\rm cr}$ make a discrete set. If extended to the supercritical region $\alpha > \alpha_{\rm cr}$, this procedure may be thought of to be equivalent to the approach recently developed by one of the present authors [13] wherein equation (15) is to be represented - by bringing the singular term $-8(\alpha/\pi s^2)\Psi$ to the right-hand side, and taking the term $-m^2\Psi$ to the left-hand side - as the generalized eigenvalue problem for the differential operator $-\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{3}{4s^2} + m^2$ defining the spectrum of α. This operator is self-adjoint in the (rigged) Hilbert space of functions, orthonormalizable *with the singular measure* $s^{-2}\mathrm{d}s$ and subjected to the boundary condition

$$\Psi(s_0) = 0 \qquad (19)$$

imposed at the lower edge $s = s_0$ of a normalization box. As long as s_0 is finite we face a discrete spectrum of α. The eigen-solutions are standing waves at the lower edge of the box and decrease at $s \to \infty$ like $\exp(-ms)$. In the limit $s_0 \to 0$ the levels condense to make a *continuum* of states in the supercritical region $\alpha > \alpha_{\rm cr}$. The norm of the state vector calculated with the singular measure‡ diverges in this limit,

‡ The requirement of square integrability with the above singular measure, when extended back to the values $\alpha < \alpha_{\rm cr}$ ($s_0 = 0$ in this case), just excludes the lower sign in (16) and makes the imposing of the conditions (17) discussed above for the bound state problem unnecessary.

what makes it possible to normalize the solution to δ-function and hence interpret it as corresponding to a free particle living mostly near the singularity. This situation refers to the kinematical domain called sector III in [13]. The corresponding solutions were called confined states, since their wave-function decreases at infinity like that of bound states. Near the origin there are free particles, incoming from the origin, then scattered elastically inwards and then outgoing back to the origin.

When applied to the supercritical nucleus [5], this approach led to the effect of absorption of electrons by the nucleus and to the known effect of electron-positron pair production, the corresponding probabilities being calculated in a cut-off-independent way. These effects, however, occurred in another kinematical domain, called sector IV, where the particles are free also at large distances from the singular center. To see, if analogous effects are intrinsic to the positronium in the supercritical case $\alpha > \alpha_{\rm cr}$ and cause an instability of the vacuum state relative to the pair production it would be necessary to go beyond the kinematical restriction $P_\mu = 0$, since we need solutions, oscillating at infinity as well as near zero to this end. Leaving this task for future, we now consider the features of the confined state.

The solution to equation (15) for $\alpha > \alpha_{\rm cr}$ that decreases when $s \to \infty$ is given by the McDonald function with imaginary index

$$\Psi(s) = \sqrt{s}\, K_{{\rm i}\sqrt{\frac{8\alpha}{\pi}-1}}(ms). \tag{20}$$

This function is real. The lower edge of the normalization box should be chosen much smaller than the characteristic length of the problem, which is the electron Compton length, $s_0 \ll m^{-1}$. Then one can use the asymptotic form of the McDonald function near zero to write the standing wave boundary condition (19) as an equation for determining the spectrum of α

$$\left(\frac{ms_0}{2}\right)^{2\nu} = \frac{\Gamma(1+\nu)}{\Gamma^*(1-\nu)}, \quad \nu = {\rm i}\sqrt{\frac{8\alpha}{\pi}-1} \tag{21}$$

or

$$\nu \ln\frac{ms_0}{2} = {\rm i}\arg\Gamma(\nu+1) - {\rm i}\pi n, \qquad n = 0, \pm1, \pm2, ... \tag{22}$$

Confining ourselves to the values of the coupling constant that do not differ much from the critical value, $|\nu| \ll 1$ we may exploit the approximation for the Euler Γ-function

$$\Gamma(1+\nu) \cong 1 - \nu C_{\rm E}, \tag{23}$$

where $C_{\rm E} = 0.577$ is the Euler constant, to get

$$\ln\left(\frac{ms_0}{2}\right) = \frac{-\pi n}{\sqrt{\frac{8\alpha_n}{\pi}-1}} - C_{\rm E}, \quad n = 1, 2, ... \tag{24}$$

We have expelled the non-positive integers n from here, since they would lead to the roots for ms_0 of the order of or much larger than unity in contradiction to the adopted condition $s_0 \ll m^{-1}$. For such values the asymptotic representation of the McDonald function used above is not valid. It may be checked that there are no other zeros of McDonald function, besides those in (24), enumerated by positive integers. Finally, the

discrete spectrum of the coupling constant above the critical value $\pi/8$ close to it is given as

$$\alpha_n = \frac{\pi}{8} + \frac{\pi}{8}\frac{n^2}{(\ln\frac{2}{ms_0} - C_{\rm E})^2}, \quad n = 1, 2, 3... \tag{25}$$

It is seen explicitly, that the eigenvalues do condense when the lower edge of the normalization box s_0 tends to zero. The wave function (20) is mostly concentrated inside the Compton radius m^{-1}, but the probability density is confined to the region close to $s = s_0 \to 0$ due to the singularity of the measure s^{-2} near this point.

The solution of the Bethe-Salpeter equation, originally written for positronium atom, relates in the ultra-relativistic situation $P_\mu = 0$ considered, as a matter of fact to the vacuum as well. Indeed, the state described has no total energy and no total momentum and correspondingly does not depend on the coordinate sums of the constituting particles. It is not separated from the vacuum by an energy barrier. Besides, it is proportional to the unit matrix in the spinor space and is O(3.1)-invariant, i.e. possesses the maximum symmetry, as the vacuum should do. On the other hand, this state has a nontrivial space-time structure, described by the dependence on the coordinate differences, which implies the concentration of the state near zero value of the relative coordinate. These considerations may mean a need of the restructuring of the vacuum when the coupling constant exceeds the critical value and serve to establish the band of values for this constant beyond which the existing standard concepts no longer hold. This is how the things stand with the positronium atom.

In the next section we derive a two-dimensional analog of equation (2) relating to the case where a strong magnetic field is imposed, and describe all solutions corresponding to the fall-down to the center. The two-dimensioning makes this phenomenon stronger. We shall return to the analysis of its consequences in the subsequent section. The important difference with the present situation will be that the agent providing the fall-down to the center will be the external magnetic field, whereas the coupling constant will be kept equal to its experimental value $\alpha = 1/137$ throughout.

3. Derivation of two-dimensional Bethe-Salpeter equation in asymptotically strong magnetic field

The view that charged particles in a strong constant magnetic field are confined to the lowest Landau level and behave effectively as if they possess only one spacial degree of freedom - the one along the magnetic field - is widely accepted. Moreover, a conjecture exists [22] that the Feynman rules in the high magnetic field limit may be directly served by two-dimensional (one space + one time) form of electron propagators. As applied to the Bethe-Salpeter equation, the dimensional reduction in high magnetic field was considered in [7], [8], [9], [11]. In these references the well-known simultaneous approximation to the Bethe-Salpeter equation taken in the integral form was exploited, appropriate for nonrelativistic treatment of the relative motion of the two charged particles. Once we shall in the next Section be interested in the ultrarelativistic regime,

we reject from using this approximation, and find it convenient to deal only with the differential form of the Bethe-Salpeter equation.

The electron-positron bound state is described by the Bethe-Salpeter amplitude (wave function) $\chi_{\alpha,\beta}(x^{\rm e},x^{\rm p})$ subject to the fully relativistic equation [15], which in the ladder approximation in a magnetic field may be written as

$$\begin{aligned}&[{\rm i}\widehat{\partial}^{\rm e}-m+e\widehat{A}(x^{\rm e})]_{\alpha\beta}[{\rm i}\widehat{\partial}^{\rm p}-m-e\widehat{A}(x^{\rm p})]_{\mu\nu}\ \ \chi_{\beta\nu}(x^{\rm e},x^{\rm p})\ \ = \\ &= -{\rm i}8\pi\alpha D_{ij}(x^{\rm e}-x^{\rm p})[\gamma_i]_{\alpha\beta}[\gamma_j]_{\mu\nu}\ \ \chi_{\beta\nu}(x^{\rm e},x^{\rm p})\end{aligned} \tag{26}$$

Here $x^{\rm e},x^{\rm p}$ are the electron and positron 4-coordinates, $D_{ij}(x^{\rm e}-x^{\rm p})$ is the photon propagator, and we have explicitly written the spinor indices $\alpha,\ \beta,\ \mu,\ \nu=1,2,3,4$.

We refer to, if needed, the so called spinor representation of the Dirac γ-matrices in the block form

$$\gamma_0=\begin{pmatrix}0 & {\rm I}\\ {\rm I} & 0\end{pmatrix},\qquad \gamma_k=\begin{pmatrix}0 & -\sigma_k\\ \sigma_k & 0\end{pmatrix}, \tag{27}$$

σ_k are the Pauli matrices:

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad {\rm i}\sigma_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{28}$$

$k=1,2,3;$ m is the electron mass, e the absolute value of its charge $e=2\sqrt{\pi\alpha}$. The metrics in the Minkowsky space is diag $g_{ij}=(1,-1,-1,-1)$. The vector potential of the constant and homogeneous magnetic field B, directed along the axis 3 ($B_3=B,\ B_{1,2}=0$), is chosen in the asymmetric gauge

$$A_1(x)=-Bx_2,\ A_{0,2,3}(x)=0. \tag{29}$$

With this choice, the translational invariance along the directions 0,1,3 holds.

Solutions to equation (26) may be represented in the form

$$\begin{aligned}&\chi(x^{\rm e},x^{\rm p})\ \ = \\ &\eta(x_0^{\rm e}-x_0^{\rm p},x_3^{\rm e}-x_3^{\rm p},x_{1,2}^{\rm e},x_{1,2}^{\rm p})\ \ \exp\{\frac{\rm i}{2}(P_0(x_0^{\rm e}+x_0^{\rm p})-P_3(x_3^{\rm e}+x_3^{\rm p}))\},\end{aligned} \tag{30}$$

where $P_{0,3}$ are the center-of-mass 4-momentum components of the longitudinal motion, that expresses the translational invariance along the longitudinal directions (0,3). We do not find it convenient to be using explicitly consequences of the magnetic translation invariance [23]. (The general representation for the Bethe-Salpeter amplitude that incorporates these features may be found in [24], [25]). Denoting the differences $x_0^{\rm e}-x_0^{\rm p}=t$, $x_3^{\rm e}-x_3^{\rm p}=z$ we obtain the equation

$$\begin{aligned}&\left[{\rm i}\hat{\partial}_\parallel-\frac{\hat{P}_\parallel}{2}-m+{\rm i}\hat{\partial}^{\rm e}_\perp-e\gamma_1A_1(x_2^{\rm e})\right]_{\alpha\beta}\left[-{\rm i}\hat{\partial}_\parallel-\frac{\hat{P}_\parallel}{2}-m+{\rm i}\hat{\partial}^{\rm p}_\perp+e\gamma_1A_1(x_2^{\rm p})\right]_{\mu\nu}\cdot \\ &\cdot\ \left[\eta(t,z,x_\perp^{\rm e,p})\right]_{\beta\nu}\ =-{\rm i}8\pi\alpha D_{ij}(t,z,x_{1,2}^{\rm e}-x_{1,2}^{\rm p})\ [\gamma_i]_{\alpha\beta}\ [\gamma_j]_{\mu\nu}\ \left[\eta(t,z,x_\perp^{\rm e,p})\right]_{\beta\nu}\ ,\end{aligned} \tag{31}$$

where $x_\perp=(x_1,x_2)$, $-\hat{\partial}_\perp=\gamma_1\partial_1+\gamma_2\partial_2$, $((\partial_\perp)_i=\partial_i,\ i=1,2)$, $\hat{\partial}_\parallel=\partial_t\gamma_0-\partial_z\gamma_3$, $\hat{P}_\parallel=P_0\gamma_0-P_3\gamma_3$.

3.1. Fourier-Ritus Expansion in eigenfunctions of the transversal motion

Expand the dependence of solution of equation (31) on the transversal degrees of freedom into the series over the (complete set of) Ritus [14] matrix eigenfunctions[+] $E_h(x_2)$

$$[\eta(t,z,x_\perp^{\rm e,p})]_{\mu\nu} \;=\sum_{h^{\rm e}h^{\rm p}} {\rm e}^{{\rm i}p_1^{\rm e}x_1^{\rm e}}[E^{\rm e}_{h^{\rm e}}(x_2^{\rm e})]_\mu^{\alpha^{\rm e}}[E^{\rm p}_{h^{\rm p}}(x_2^{\rm p})]_\nu^{\alpha^{\rm p}}{\rm e}^{{\rm i}p_1^{\rm p}x_1^{\rm p}}\quad[\eta_{h^{\rm e}h^{\rm p}}(t,z)]^{\alpha^{\rm e}\alpha^{\rm p}}\ . \tag{32}$$

Here $\eta_{h^{\rm e}h^{\rm p}}(t,z)$ denote *unknown* functions that depend on the differences of the longitudinal variables, while the Ritus matrix functions ${\rm e}^{{\rm i}p_1x_1}E_h(x_2)$ depend on the individual coordinates $x_{1,2}^{\rm e,p}$ transversal to the field. The Ritus matrix functions and the unknown functions $\eta_{h^{\rm e}h^{\rm p}}(t,z)$ are labelled by two pairs $h^{\rm e},h^{\rm p}$ of quantum numbers $h=(k,p_1,)$, each pair relating to one out of the two particles in a magnetic field. The Landau quantum number k runs all nonnegative integers $k=0,1,2,3...$, while p_1 is the particle momentum component along the transversal axis 1. Recall that the potential $A_\mu(x)$ (29) does not depend on x_1, so that p_1 does conserve. This quantum number is connected with the orbit center coordinate $\widetilde{x_2}$ along the axis 2 [1], $p_1=-\widetilde{x_2}eB$.

The matrix functions ${\rm e}^{{\rm i}p_1x_1}E_h^{\rm e,p}(x_2)$ for transverse motion in the magnetic field (29), relating in (32) to electrons (e) and positrons (p), are 4×4 matrices, formed, in the spinor representation, by four eigen-bispinors of the operator $({\rm i}\hat\partial_\perp\pm e\hat A)^2$

$$({\rm i}\hat\partial_\perp\pm e\hat A)^2_{\mu\nu}{\rm e}^{{\rm i}p_1x_1}[E_h^{\rm e,p}(x_2)]_\nu^{(\sigma,\gamma)}=-2eBk{\rm e}^{{\rm i}p_1x_1}[E_h^{\rm e,p}(x_2)]_\mu^{(\sigma,\gamma)}, \tag{33}$$

placed, as columns, side by side [14]. Here the upper and lower signs relate to electron and positron, respectively, while $\sigma=\pm1$ and $\gamma=\pm1$ are eigenvalues of the operators

$$\Sigma_3=\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix},\qquad -{\rm i}\gamma_5=\begin{pmatrix}\text{-I} & 0\\ 0 & \text{I}\end{pmatrix}, \tag{34}$$

diagonal in the spinor representation, to which the same 4-spinors are eigen-bispinors[*]

$$-\,{\rm i}\gamma_5E_h^{(\sigma,\gamma)}=\gamma E_h^{(\sigma,\gamma)},\quad \Sigma_3E_h^{(\sigma,\gamma)}=\sigma E_h^{(\sigma,\gamma)}.\quad sg \tag{35}$$

The couple of indices $\alpha=(\sigma,\gamma)$ is united into one index α in the expansion (32), $\alpha=1,2,3,4$ according to the convention: $(+1,-1)=1,\ \ (-1,-1)=2,\ (+1,+1)=3,\ (-1,+1)=4$. With this convention, the set of 4-spinors $[E_h(x_2]_\mu^{(\sigma,\gamma)}=E_h(x_2)_\mu^\alpha$ can be dealt with as a 4×4 matrix, the united index α spanning a matrix space, where the usual algebra of γ -matrices may act. Correspondingly, in (32) the unknown function $\left[\eta_{h^{\rm e}h^{\rm p}}(t,z)\right]^{\alpha^{\rm e}\alpha^{\rm p}}$ is a matrix in the same space, and contracts with the Ritus matrix function.

Following [14], the matrix functions in expansion (32) can be written in the block form as diagonal matrices

$${\rm e}^{{\rm i}p_1x_1}E_h^{\rm e,p}(x_2)=\begin{pmatrix}a^{\rm e,p}(h;x_{1,2}) & 0\\ 0 & a^{\rm e,p}(h;x_{1,2})\end{pmatrix},$$

[+] Our definition of the matrix eigenfunctions differs from that of Ref. [14] in that the longitudinal degrees of freedom are not included and the factor $\exp({\rm i}p_1x_1)$ is separated.

[*] Henceforth, if superscripts e or p are omitted, the corresponding equations relate both to electrons and positrons in a form-invariant way.

$$a^{\rm e,p}(h;x_{1,2}) = \left(\begin{array}{cc} a^{\rm e,p}_{+1}(h;x_{1,2}) & 0 \\ 0 & a^{\rm e,p}_{-1}(h;x_{1,2}) \end{array} \right). \tag{36}$$

Here $a^{\rm e,p}_{\sigma}(h;x_{1,2})$ are eigenfunctions of the two (for each sign $\pm$) operators $[((\mathrm{i}\partial_\perp)_\alpha \pm eA_\alpha)^2 \mp \sigma eB]$, labelled by the two values $\sigma = 1, \ -1$

$$[-\left((\mathrm{i}\partial_\perp)_\alpha \pm eA_\alpha\right)^2 \pm \sigma eB] a^{\rm e,p}_{\sigma}(h;x_{1,2}) = -2eBk a^{\rm e,p}_{\sigma}(h;x_{1,2}), \tag{37}$$

Namely, (we omit the subscript "1" by p_1 in what follows)

$$a^{\rm e,p}_{\sigma}(h;x_{1,2}) = \mathrm{e}^{\mathrm{i}px_1} U_{k+\frac{\pm\sigma-1}{2}}\left(\sqrt{eB}\left(x_2 \pm \frac{p}{eB}\right)\right), \quad k = 0,1,2..., \tag{38}$$

with

$$U_n(\xi) = \exp\left\{-\frac{\xi^2}{2}\right\}(2^n n!\sqrt{\pi})^{-\frac{1}{2}} H_n(\xi) \tag{39}$$

being the normalized Hermite functions ($H_n(\xi)$ are the Hermite polynomials). Equations (37) are the same as (33) due to the relation

$$(\mathrm{i}\hat{\partial}_\perp \pm e\hat{A})^2 = -\left((\mathrm{i}\partial_\perp)_\alpha \pm eA_\alpha\right)^2 \pm eB\Sigma_3 \tag{40}$$

and to eq.(35). Besides, the matrix functions (36) are eigenfunctions to the operator $-\mathrm{i}\partial_1$, as commuting with Σ_3 and γ_5 (34), and with $(\mathrm{i}\hat{\partial}_\perp + e\hat{A})^2_{\mu\nu}$. The corresponding eigenvalue p_1 does not, however, appear in the r.-h. side of (37) due to the well-known degeneracy of electron spectrum in a constant magnetic field.

The orthonormality relation for the Hermite functions

$$\int_{-\infty}^{\infty} U_n(\xi)U_{n'}(\xi)\mathrm{d}\xi = \delta_{nn'}. \tag{41}$$

implies the orthogonality of the Ritus matrix eigenfunctions in the form

$$\sqrt{eB}\int E^*_h(x_2)^\alpha_\mu E_{h'}(x_2)^{\alpha'}_\mu \mathrm{d}x_2 = \delta_{kk'}\delta_{\alpha\alpha'}. \tag{42}$$

As a matter of fact, the matrix functions $E_h(x_2)$ are real, and we henceforth omit the complex conjugation sign "*".

The matrix functions $\mathrm{e}^{\mathrm{i}px_1}E^{\rm e,p}_h(x_2)$ (36) commute with the longitudinal part $\pm\,\mathrm{i}\hat{\partial}_\parallel - \hat{P}_\parallel/2 - m$ of the Dirac operator in (31), owing to the commutativity property

$$[E_h(x_2), \gamma_{0,3}]_- = 0, \tag{43}$$

and are [14], in a sense, matrix eigenfunctions of the transversal part of Dirac operator (not only of its square (33))

$$(\mathrm{i}\hat{\partial}_\perp \pm e\hat{A})\mathrm{e}^{\mathrm{i}px_1}E^{\rm e,p}_h(x_2) = \pm\sqrt{2eBk}\ \mathrm{e}^{\mathrm{i}px_1}E^{\rm e,p}_h(x_2)\gamma_1. \tag{44}$$

The Landau quantum number k appears here as a "universal eigenvalue" thanks to the mechanism, easy to trace, according to which the differential operator in the left-hand side of eq.(44) acts as a lowering or rising operator on the functions (39), whereas the matrix σ_2, involved in γ_2, interchanges the places the functions U_k, U_{k-1} occupy in the columns. Contrary to relations, which explicitly include the variable σ, whose value forms the number of the corresponding column, relations (33), (44), (43), and the first relation in (35) are covariant with respect to passing to other representation of γ-matrices, where the matrix $E_h(x_2)$ may become non-diagonal.

3.2. Equation for the Fourier-Ritus transform of the Bethe-Salpeter amplitude.

Now we are in a position to use expansion (32) in equation (31). We left multiply it by $(2\pi)^{-2}eB\ \mathrm{e}^{-\mathrm{i}\overline{p}^{\mathrm{e}}x_1^{\mathrm{e}}}E^{\mathrm{e}}_{\overline{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\mathrm{e}^{-\mathrm{i}\overline{p}^{\mathrm{p}}x_1^{\mathrm{p}}}E^{\mathrm{p}}_{\overline{h}^{\mathrm{p}}}(x_2^{\mathrm{p}})$, then integrate over $\mathrm{d}^2x^{\mathrm{e}}_{1,2}\ \mathrm{d}^2x^{\mathrm{p}}_{1,2}$. After using (44) and (43), and exploiting the orthonormality relation (42) for the summation over the quantum numbers $h^{\mathrm{e,p}} = (k^{\mathrm{e,p}}, p_1^{\mathrm{e,p}})$, the following expression :

$$\left[\mathrm{i}\hat{\partial}_{\|} - \frac{\hat{P}_{\|}}{2} - m + \gamma_1\sqrt{2eBk^{\mathrm{e}}}\right]_{\alpha\alpha^{\mathrm{e}}}\left[-\mathrm{i}\hat{\partial}_{\|} - \frac{\hat{P}_{\|}}{2} - m - \gamma_1\sqrt{2eBk^{\mathrm{p}}}\right]_{\mu\alpha^{\mathrm{p}}}\quad[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]^{\alpha^{\mathrm{e}}\alpha^{\mathrm{p}}} \tag{45}$$

is obtained for the left-hand side of the Fourier-Ritus-transformed equation (31). We omitted the bars over the quantum numbers.

Taking the expression

$$D_{ij}\left(t,z,x^{\mathrm{e}}_{1,2}-x^{\mathrm{p}}_{1,2}\right) = \frac{g_{ij}}{\mathrm{i}4\pi^2}\left(t^2 - z^2 - (x_1^{\mathrm{e}}-x_1^{\mathrm{p}})^2 - (x_2^{\mathrm{e}}-x_2^{\mathrm{p}})^2\right)^{-1}, \tag{46}$$

for the photon propagator in the Feynman gauge, we may then write the right-hand side of Ritus-transformed equation (31) as

$$\frac{\alpha}{2\pi^3}\int \mathrm{d}p^{\mathrm{e}}\,\mathrm{d}p^{\mathrm{p}}\sum_{k^{\mathrm{e}}k^{\mathrm{p}}} g_{ij}\int [E^{\mathrm{e}}_{\overline{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\gamma_i E^{e}_{h^{\mathrm{e}}}(x_2^{e})]_{\alpha\alpha_{\mathrm{e}}}\ [E^{p}_{\overline{h}^{p}}(x_2^{p})\gamma_j E^{\mathrm{p}}_{h^{\mathrm{p}}}(x_2^{\mathrm{p}})]_{\mu\alpha^{\mathrm{p}}}\ [\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]^{\alpha^{\mathrm{e}}\alpha^{\mathrm{p}}}$$
$$\frac{\mathrm{e}^{\mathrm{i}(p^{\mathrm{e}}-\overline{p}^{\mathrm{e}})x_1}\ \mathrm{e}^{\mathrm{i}(p^{\mathrm{p}}-\overline{p}^{\mathrm{p}})x_1}\ eB\mathrm{d}^2x^{e}_{1,2}\ \mathrm{d}^2x^{p}_{1,2}}{z^2 + (x_1^{\mathrm{e}}-x_1^{\mathrm{p}})^2 + (x_2^{\mathrm{e}}-x_2^{\mathrm{p}})^2 - t^2}, \tag{47}$$

Integrating explicitly the exponentials in (47) over the variable $X = (x_1^{\mathrm{e}} + x_1^{\mathrm{p}})/2$, we obtain the following expression:

$$\frac{\alpha}{\pi^2}\int \mathrm{d}p\ \mathrm{d}P_1\ \delta(\overline{P}_1 - P_1)\sum_{k^e k^p} g_{ij}\int [E^{\mathrm{e}}_{\overline{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\gamma_i E^{e}_{h^{\mathrm{e}}}(x_2^{e})]_{\alpha\alpha_{\mathrm{e}}}\ [E^{p}_{\overline{h}^{p}}(x_2^{p})\gamma_j E^{\mathrm{p}}_{h^{\mathrm{p}}}(x_2^{\mathrm{p}})]_{\mu\alpha^{\mathrm{p}}}\cdot$$
$$[\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]^{\alpha^{\mathrm{e}}\alpha^{\mathrm{p}}}\ \frac{\exp(\mathrm{i}x(\overline{p}-p))\mathrm{d}x}{z^2 + x^2 + (x_2^{\mathrm{e}}-x_2^{\mathrm{p}})^2 - t^2}\ eB\mathrm{d}x_2^{\mathrm{e}}\mathrm{d}x_2^{\mathrm{p}}, \tag{48}$$

where the new integration variables $x = x_1^{\mathrm{e}} - x_1^{\mathrm{p}}$, $P_1 = p^{\mathrm{e}} + p^{\mathrm{p}}$, $p = (p^{\mathrm{e}} - p^{\mathrm{p}})/2$ and the new definitions $\overline{P}_1 = \overline{p}^{\mathrm{e}} + \overline{p}^{\mathrm{p}}$, $\overline{p} = (\overline{p}^{\mathrm{e}} - \overline{p}^{\mathrm{p}})/2$ have been introduced. The pairs of quantum numbers in (48) are

$$\overline{h}^{\mathrm{e,p}} = (\overline{k}^{\mathrm{e,p}}, \frac{\overline{P}_1}{2} \pm \overline{p}), \qquad h^{\mathrm{e,p}} = (k^{\mathrm{e,p}}, \frac{P_1}{2} \pm p). \tag{49}$$

Hence the arguments of the functions (38) in (48) are:

$$\sqrt{eB}\left(x_2^{\mathrm{e}} + \frac{\overline{P}_1 + 2\overline{p}}{2eB}\right),\quad \sqrt{eB}\left(x_2^{\mathrm{e}} + \frac{P_1 + 2p}{2eB}\right),\quad \sqrt{eB}\left(x_2^{\mathrm{p}} - \frac{\overline{P}_1 - 2\overline{p}}{2eB}\right), \sqrt{eB}\left(x_2^{\mathrm{p}} - \frac{P_1 - 2p}{2eB}\right), \tag{50}$$

successively as the functions $E_h(x_{1,2})$ appear in (48) from left to right. After fulfilling the integration over $\mathrm{d}P_1$ with the use of the δ-function, introduce the new integration variable $q = p - \overline{p}$ instead of p, and the integration variables $\overline{x}_2^{\mathrm{e}} = x_2^{\mathrm{e}} + (\overline{P}_1 + 2\overline{p})/2eB$, $\overline{x}_2^{\mathrm{p}} = x_2^{\mathrm{p}} - (\overline{P}_1 - 2p)/2eB$ instead of x_2^{e} and x_2^{p}. Then eq.(48) may be written as

$$\frac{\alpha}{\pi^2}\int \mathrm{d}q\sum_{k^{\mathrm{e}}k^{p}} g_{ij}\int [E^{\mathrm{e}}_{\overline{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\gamma_i E^{e}_{h^{\mathrm{e}}}(x_2^{e})]_{\alpha\alpha_{\mathrm{e}}}\ [E^{p}_{\overline{h}^{p}}(x_2^{p})\gamma_j E^{\mathrm{p}}_{h^{\mathrm{p}}}(x_2^{\mathrm{p}})]_{\mu\alpha^{\mathrm{p}}}\ [\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]^{\alpha^{\mathrm{e}}\alpha^{\mathrm{p}}}\ \cdot$$

$$\int \frac{\exp(-\mathrm{i}xq)\ \mathrm{d}x\ eB\ \mathrm{d}\overline{x}_2^e\ \mathrm{d}\overline{x}_2^p}{z^2+x^2+\left(\overline{x}_2^{\mathrm{e}}-\overline{x}_2^{\mathrm{p}}-\frac{\overline{P}_1-q}{eB}\right)^2-t^2} \tag{51}$$

Now the pairs of quantum numbers in (51) are

$$\overline{h}^{\mathrm{e,p}}=(\overline{k}^{\mathrm{e,p}},\frac{\overline{P}_1}{2}\pm\overline{p}),\qquad h^{\mathrm{e,p}}=(k^{\mathrm{e,p}},\frac{\overline{P}_1}{2}\pm q\pm\overline{p}). \tag{52}$$

Hence the arguments of the functions (38) in (51) from left to right are:

$$\sqrt{eB}\overline{x}_2^{\mathrm{e}},\quad \sqrt{eB}\left(\overline{x}_2^{\mathrm{e}}+\frac{q}{eB}\right),\quad \sqrt{eB}\left(\overline{x}_2^{\mathrm{p}}-\frac{q}{eB}\right),\quad \left(\sqrt{eB}\overline{x}_2^{\mathrm{p}}\right). \tag{53}$$

3.3. Adiabatic approximation.

Now we aim at passing to the large magnetic field regime in the Bethe-Salpeter equation, with (45) as the left-hand side and (51) as the right-hand side. Define the dimensionless integration variables $w=x\sqrt{eB}$, $q'=q/\sqrt{eB}$, $\xi^{\mathrm{e,p}}=\overline{x}_2^{\mathrm{e,p}}\sqrt{eB}$ in eq.(51). Then it takes the form

$$\frac{\alpha}{\pi^2}\int \mathrm{d}q' \sum_{k^{\mathrm{e}}k^{\mathrm{p}}} g_{ij} \int [E^{\mathrm{e}}_{\overline{h}^{\mathrm{e}}}(x_2^{\mathrm{e}})\gamma_i E^{\mathrm{e}}_{h^{\mathrm{e}}}(x_2^{e})]_{\alpha\alpha_{\mathrm{e}}}\ [E^{\mathrm{p}}_{\overline{h}^{\mathrm{p}}}(x_2^{\mathrm{p}})\gamma_j E^{\mathrm{p}}_{h^{\mathrm{p}}}(x_2^{\mathrm{p}})]_{\mu\alpha^{\mathrm{p}}}\ [\eta_{h^{\mathrm{e}}h^{\mathrm{p}}}(t,z)]^{\alpha^{\mathrm{e}}\alpha^{\mathrm{p}}}\ \cdot$$

$$\cdot\int \frac{\exp(-\mathrm{i}wq')\mathrm{d}w\mathrm{d}\xi^{\mathrm{e}}\mathrm{d}\xi^{\mathrm{p}}}{z^2+\frac{w^2}{eB}+\frac{1}{eB}\left(\xi^{\mathrm{e}}-\xi^{\mathrm{p}}-\frac{\overline{P}_1}{\sqrt{eB}}-q'\right)^2-t^2}. \tag{54}$$

The pairs of quantum numbers in (54) are

$$\overline{h}^{\mathrm{e,p}}=(\overline{k}^{\mathrm{e,p}},\frac{\overline{P}_1}{2}\pm\overline{p}),\qquad h^{\mathrm{e,p}}=(k^{\mathrm{e,p}},\frac{\overline{P}_1}{2}\pm q'\sqrt{eB}\pm\overline{p}). \tag{55}$$

The arguments of the functions (38) in (54) from left to right are:

$$\xi^{\mathrm{e}},\quad \xi^{\mathrm{e}}+q',\quad \xi^{\mathrm{p}}-q',\quad \xi^{\mathrm{p}}. \tag{56}$$

When considering the large field behavior we admit for completeness that the difference between the centers of orbits along the axis 2 $\widetilde{x}_2^{\mathrm{e}}-\widetilde{x}_2^{\mathrm{p}}=-\frac{\overline{P}_1}{eB}$ may be kept finite, in other words that the transversal momentum $\overline{P}_1$ grows linearly with the field. We shall see that that big transversal momenta do not contradict dimensional compactification, but produce an extra regularization of the light-cone singularity.

In the region, where the 2-interval $(z^2-t^2)^{1/2}$ essentially exceeds the Larmour radius $L_B=1/\sqrt{eB}$,

$$z^2-t^2\gg L_B^2 \tag{57}$$

one may neglect the dependence on the integration variables w and later on $\xi_{\mathrm{e,p}}$ in the denominator. Integration over w produces $2\pi\delta(q')$, which annihilates the dependence on q' in the arguments (53) of the Hermite functions, and they all equalize.

Let us depict this mechanism in more detail. Fulfill explicitly the integration over dw in (54):

$$\int\frac{\exp(-\mathrm{i}wq')\ \mathrm{d}w}{z^2-t^2+\frac{w^2}{eB}+\frac{A^2}{eB}}=\frac{\sqrt{eB}\pi}{\sqrt{z^2-t^2+\frac{A^2}{eB}}}\left(\theta(q')\exp\left(-q'\sqrt{eB(z^2-t^2)+A^2}\ \right)+\right.$$

$$\theta(-q')\exp\Big(\,q'\sqrt{eB(z^2-t^2)+A^2}\,\Big)\Big), \tag{58}$$

where

$$A^2=\left(\xi^e-\xi^p-\frac{\overline{P}_1}{\sqrt{eB}}-q'\right)^2 \tag{59}$$

and $\theta(q')$ is the step function,

$$\theta(q')=\begin{cases}1 & \text{when}\quad q'>0,\\ \frac{1}{2} & \text{when}\quad q'=0,\\ 0 & \text{when}\quad q'<0.\end{cases} \tag{60}$$

Due to the decreasing exponential in (39) the variables $\xi^{\rm e,p}$ do not exceed unity in the order of magnitude and can be neglected as compared to $\frac{\overline{P}_1}{\sqrt{eB}}$ in (59). Unless q' is large it may be neglected as compared to the same term in (59), too. Then $A^2=\frac{\overline{P}_1^2}{eB}$, and after (58) is substituted in (54) and integrated over dq' the contribution comes only from the integration within the shrinking region $|q'|<(eB[z^2-t^2+\frac{\overline{P}_1^2}{(eB)^2}])^{-\frac{1}{2}}$. Then q' can be also neglected in the arguments (56). If, contrary to the previous assumption, we admit that $|q'|$ is of the order of $\frac{\overline{P}_1}{\sqrt{eB}}\sim\sqrt{eB}$ we see that the exponentials in (58) fast decrease with the growth of the magnetic field as $\exp(-eB(z^2-t^2))$, and therefore such values of $|q'|$ do not contribute to the integration. If we admit, last, that $|q'|\gg|\frac{\overline{P}_1}{\sqrt{eB}}|$, we find that the contribution $\exp(-|q'|\sqrt{eB(z^2-t^2)+(q')^2}\,)$ from the integration over such values is still smaller. Thus, we have justified the possibility to omit the dependence on q' in (59) and in (56), and also on $\xi^{\rm e,p}$ in (59). Now we can perform the integration over dq' to obtain the following expression for (54)

$$\frac{2\alpha\pi^{-1}}{z^2+\frac{P_1^2}{(eB)^2}-t^2}\cdot$$

$$\cdot\sum_{k^{\rm e}k^{\rm p}}g_{ii}\int[E^{\rm e}_{\overline{h}^{\rm e}}(x_2^{\rm e})\gamma_iE^{\rm e}_{h^{\rm e}}(x_2^{\rm e})]_{\alpha\alpha_{\rm e}}{\rm d}\xi^{\rm e}\int[E^{\rm p}_{\overline{h}^{\rm p}}(x_2^{\rm p})\gamma_iE^{\rm p}_{h^{\rm p}}(x_2^{\rm p})]_{\mu\alpha^{\rm p}}{\rm d}\xi^{\rm p}\;[\eta_{h^{\rm e}h^{\rm p}}(t,z)]^{\alpha^{\rm e}\alpha^{\rm p}} \tag{61}$$

It remains yet to argue that the limit (61) is valid also when the term $\frac{\overline{P}_1}{eB}$ is not kept. In this case we no longer can disregard q' inside A^2 when q' is less than or of the order of unity. But we can disregard A^2 as compared with $eB(z^2-t^2)$ to make sure that the integration over dq' is restricted to the region close to zero $|q|'<(eB(z^2-t^2))^{-1/2}$ and hence set $q'=0$ in (56). The contribution of large q' is small as before.

The integration over $\xi_{\rm e,p}$ of the terms with $i=0,3$ in (61) yields the Kroneker deltas $\delta_{k^{\rm e}\overline{k}^{\rm e}}\;\delta_{k^{\rm p}\overline{k}^{\rm p}}$ due to the orthonormality (41) of the Hermite functions thanks to the commutativity (43) of the Ritus matrix functions (36) with γ_0 and γ_3. On the contrary, $\gamma_1,\ \gamma_2$ do not commute with (36). This implies the appearance of terms, non-diagonal in Landau quantum numbers, like $\delta_{k^{\rm e},\overline{k}^{\rm e}\pm1}$ and $\delta_{k^{\rm p},\overline{k}^{\rm p}\pm1}$, in (54), proportional to $(i=1,2)$:

$$T^i_{\overline{k}_{\rm e}\pm1,\overline{k}^{\rm p}\pm1}=\sum_{k^{\rm e}k^{\rm p}}\int[E^{\rm e}_{\overline{h}^{\rm e}}(x_2^{\rm e})\gamma_iE^{\rm e}_{h^{\rm e}}(x_2^{\rm e})]_{\alpha\alpha_{\rm e}}{\rm d}\xi^{\rm e}\int[E^{\rm p}_{\overline{h}^{\rm p}}(x_2^{\rm p})\gamma_iE^{\rm p}_{h^{\rm p}}(x_2^{\rm p})]_{\mu\alpha^{\rm p}}{\rm d}\xi^{\rm p}\;[\eta_{h^{\rm e}h^{\rm p}}(t,z)]^{\alpha^{\rm e}\alpha^{\rm p}}\;=$$

$$=\sum_{k^{\rm e}k^{\rm p}}\begin{pmatrix}0 & -\Delta^i_{\overline{k}^{\rm e}k^{\rm e}}\\ \Delta^i_{\overline{k}^{\rm e}k^{\rm e}} & 0\end{pmatrix}_{\alpha\alpha^{\rm e}}\begin{pmatrix}0 & -\Delta^i_{\overline{k}^{\rm p}k^{\rm p}}\\ \Delta^i_{\overline{k}^{\rm p}k^{\rm p}} & 0\end{pmatrix}_{\mu\alpha^{\rm p}}[\eta_{h^{\rm e}h^{\rm p}}(t,z)]^{\alpha^{\rm e}\alpha^{\rm p}}\;. \tag{62}$$

Here $x_2^{\rm e,p}$ are expressed in terms of ξ through the chain of the changes of variables made above starting from (47), so that all the arguments of the Hermite functions have become equal to ξ. Besides,

$$h^{\rm e,p} = (k^{\rm e,p}, \overline{p}^{\rm e,p}), \qquad \overline{h}^{\rm e,p} = (\overline{k}^{\rm e,p}, \overline{p}^{\rm e,p}), \quad p^{\rm e} + p^{\rm p} = P_1. \tag{63}$$

$$\Delta^i_{\overline{k}k} = \int a'(\overline{h}, x_2)\ \sigma_i\ a'(h, x_2) {\rm d}\xi, \qquad i = 1, 2 \tag{64}$$

$$\Delta^{(1)}_{\overline{k}k} = \int \left(\begin{array}{cc} 0 & a'_{+1}(\overline{h}, x_2) a'_{-1}(k, x_2) \\ a'_{-1}(\overline{h}, x_2) a'_{+1}(h, x_2) & 0 \end{array} \right) {\rm d}\xi =$$
$$= \left(\begin{array}{cc} 0 & \delta_{\overline{k},\,k-1} \\ \delta_{\overline{k},\,k+1} & 0 \end{array} \right), \tag{65}$$

$$\Delta^{(2)}_{\overline{k}k} = {\rm i} \int \left(\begin{array}{cc} 0 & -a'_{+1}(\overline{h}, x_2) a'_{-1}(k, x_2) \\ a'_{-1}(\overline{h}, x_2) a'_{+1}(h, x_2) & 0 \end{array} \right) {\rm d}\xi =$$
$$= {\rm i} \left(\begin{array}{cc} 0 & -\delta_{\overline{k},\,k-1} \\ \delta_{\overline{k},\,k+1} & 0 \end{array} \right), \tag{66}$$

The prime over a indicates that the exponential $\exp({\rm i}px_1)$ is dropped from the definitions (36) and (38). The non-diagonal Kronecker deltas appeared, because $a'_{\pm 1}(\overline{h}, x_2)$ are multiplied by $a'_{\mp 1}(h, x_2)$ under the action of the $\sigma_{1,2}$-blocks in $\gamma_{1,2}$ (27). In the final form, the matrices in (62) are

$$\left(\begin{array}{cc} 0 & -\Delta^i_{\overline{k}k} \\ \Delta^i_{\overline{k}k} & 0 \end{array} \right) = \frac{1}{2} \left(\gamma_1 (\pm \delta_{\overline{k},k-1} + \delta_{\overline{k},k+1}) + {\rm i}\gamma_2 (\pm \delta_{\overline{k},k-1} - \delta_{\overline{k},k+1}) \right), \tag{67}$$

with the upper sign relating to $i = 1$ and the lower one to $i = 2$. Now equation (31) acquires the following form,

$$\left[{\rm i}\hat{\partial}_\parallel - \frac{\hat{P}_\parallel}{2} - m + \gamma_1 \sqrt{2eBk^{\rm e}} \right]_{\alpha\alpha^{\rm e}} \left[-{\rm i}\hat{\partial}_\parallel - \frac{\hat{P}_\parallel}{2} - m - \gamma_1 \sqrt{2eBk^{\rm p}} \right]_{\mu\alpha^{\rm p}} [\eta_{h^{\rm e}h^{\rm p}}(t, z)]^{\alpha^{\rm e}\alpha^{\rm p}} =$$
$$= \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \left(\sum_{i=0,3} g_{ii} [\gamma_i]_{\alpha\alpha^{\rm e}} [\gamma_i]_{\mu\alpha^{\rm p}}\ [\eta_{h^{\rm e}h^{\rm p}}(t, z)]^{\alpha^{\rm e}\alpha^{\rm p}} - \sum_{i=1,2} T^{(i)}_{k^{\rm e}\pm 1,\, k^{\rm p}\pm 1} \right), \quad p_1^{\rm e} + p_1^{\rm p} = P_1 \tag{68}$$

The bars over quantum numbers are omitted. This equation is degenerate with respect to the difference of the electron and positron momentum components $p = (p^{\rm e} - p^{\rm p})/2$ across the magnetic field, but does depend on its transversal center-of-mass momentum $P_1 = (p^{\rm e} + p^{\rm p})$. This dependence is present, however, only for sufficiently large transverse momenta P_1.

At the present step of adiabatic approximation we have come, for high magnetic field, to the chain of equations (68), in which the unknown function for a given pair of Landau quantum numbers $k^{\rm e}, k^{\rm p}$ is tangled with the same function with the Landau quantum numbers both shifted by ± 1 (in contrast to the general case of a moderate magnetic field, where these numbers may be shifted by all positive and negative integers).

To be more precise, the chain consists of two mutually disentangled sub-chains. The first one includes all functions with the Landau quantum numbers $k^{\rm e}$, $k^{\rm p}$ both even or both odd, and the second includes their even-odd and odd-even combinations. We discuss the first sub-chain since it contains the lowest function with $k^{\rm e}=k^{\rm p}=0$. We argue now that there exists a solution to the first sub-chain of equations (68), for which all $\eta_{k^{\rm e},p_1^{\rm e};\,k^{\rm p},p_1^{\rm p}}(t,z)$ disappear if at least one of the quantum numbers $k^{\rm e},k^{\rm p}$ is different from zero. Indeed, for $k^{\rm e}=k^{\rm p}=0$ equation (68) then reduces to the closed set

$$[{\rm i}\hat{\partial}_\parallel-\frac{\hat{P}_\parallel}{2}-m]_{\alpha\alpha^{\rm e}}[-{\rm i}\hat{\partial}_\parallel-\frac{\hat{P}_\parallel}{2}-m]_{\mu\alpha^{\rm p}}\ \ [\eta_{0,p_1^{\rm e};0,p_1^{\rm p}}(t,z)]^{\alpha^{\rm e}\alpha^{\rm p}}\ \ =$$

$$=\frac{2\alpha\pi^{-1}}{z^2+\frac{P_1^2}{(eB)^2}-t^2}\sum_{i=0,3}g_{ii}[\gamma_i]_{\alpha\alpha^{\rm e}}[\gamma_i]_{\mu\alpha^{\rm p}}\ \ [\eta_{0,p_1^{\rm e};0,p_1^{\rm p}}(t,z)]^{\alpha^{\rm e}\alpha^{\rm p}}\ \ ,\quad p_1^{\rm e}+p_1^{\rm p}=P_1. \tag{69}$$

In writing it we have returned to the initial designation of the electron and positron transverse momenta $p_1^{\rm e,p}$. Denote for simplicity $\eta_{k^{\rm e}k^{\rm p}}=\eta_{k^{\rm e},p_1^{\rm e};k^{\rm p},p_1^{\rm p}}(t,z)$. If we consider equation (68) with $k^{\rm e}=k^{\rm p}=1$ for η_{11} we shall have a nonzero contribution in the right-hand side, proportional to η_{00} coming from $T^i_{k^{\rm e}-1,k^{\rm p}-1}$, since the other contributions η_{11},η_{22},η_{20}, η_{02} are vanishing according to the assumption. As the left-hand side of equation (68) now contains a term, infinitely growing with the magnetic field B, it can be only satisfied with the function η_{11}, infinitely diminishing with B in the domain (57) as

$$[\eta_{11}]^{\alpha\mu}\ \ =-\frac{1}{2eB}\frac{\alpha\pi^{-1}}{z^2+\frac{P_1^2}{(eB)^2}-t^2}\sum_{i=0,3}g_{ii}[\gamma_1\gamma_i]_{\alpha\alpha^{\rm e}}[\gamma_1\gamma_i]_{\mu\alpha^{\rm p}}\ \ [\eta_{00}]^{\alpha^{\rm e}\alpha^{\rm p}}\ \ , \tag{70}$$

in accord with the assumption made. Thus, the assumption that all Bethe-Salpeter amplitudes with nonzero Landau quantum numbers are zero in the large-field case is consistent. We state that a solution to the closed set (69) for $\eta_{0,p_1^{\rm e};0,p_1^{\rm p}}(t,z)$ with all the other components equal to zero is a solution to the whole chain (68).

The derivation given in this Subsection realizes formally the known heuristic argument that, for high magnetic field, the spacing between Landau levels is very large and hence the particles taken in the lowest Landau state remain in it. Effectively, only the longitudinal degree of freedom survives for large B, the space-time reduction taking place. Equation (69) is a fully relativistic two-dimensional set of equations with two space-time arguments t and z and two gamma-matrices γ_0 and γ_3 involved. Since, unlike the previous works [7], [9], [11], neither the famous equal-time Ansatz for the Bethe-Salpeter amplitude [15], nor any other assumption concerning the non-relativistic character of the internal motion inside the positronium atom was made, the equation derived is valid for arbitrary strong binding. It will be analyzed for the extreme relativistic case in the next Subsection.

The two-dimensional equation (69) is valid in the space-like domain (57). It is meaningful provided that its solution is concentrated in this domain. In non-relativistic or semi-relativistic consideration it is often accepted that the wave function is concentrated within the Bohr radius $a_0=(\alpha m)^{-1}=0.5\times10^{-8}$ cm. It is then estimated that the corresponding analog of asymptotic equation (69) holds true when $a_0\gg L_B$,

i.e. for the magnetic fields larger than $\alpha^2 m^2/e \sim 2.35 \times 10^9$ Gauss. This estimate, however, cannot be universal and may be applicable at the most to the magnetic fields close to the lower bound where the value of the Bohr radius can be borrowed from the theory without the magnetic field. Generally, the question, where the wave function is concentrated, should be answered *a posteriori* by inspecting a solution to equation (69). Therefore, one can state, how large the fields should be in order that the asymptotic equation (69) might be trusted, no sooner that its solution is investigated. We shall return to this point when we deal with the ultra-relativistic situation.

Remind that the transverse total momentum component of the positronium system is connected with the separation between the centers of orbits of the electron and positron $P_1/(eB) = \widetilde{x}_2^{\mathrm{e}} - \widetilde{x}_2^{p}$ in the transversal plane, so that the "potential" factor in eq. (69) may be expressed in the following interesting form

$$\frac{\alpha}{(x_0^{\mathrm{e}} - x_0^{\mathrm{p}})^2 - (x_3^{\mathrm{e}} - x_3^{\mathrm{p}})^2 - (\widetilde{x}_2^{\mathrm{e}} - \widetilde{x}_2^{\mathrm{p}})^2}, \tag{71}$$

(*cf* the corresponding form of the Coulomb potential in the semi-relativistic treatment of the Bethe-Salpeter equation in [8], [9], [11]- the difference between the potentials in [9], [8] and [11] lies within the accuracy of the adiabatic approximation). The appearance of P_1^2 in the potential determines the energy spectrum dependence upon the momentum of motion of the two-particle system across the magnetic field like in [8], [9], [11], [26].

We shall need equation (69) in a more convenient form. First, transcribe it as

$$\begin{aligned}&(\mathrm{i}\overrightarrow{\hat{\partial}}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m)\ \eta_{0,p_1^{\mathrm{e}};0,p_1^{\mathrm{p}}}(t,z)\ (-\mathrm{i}\overleftarrow{\hat{\partial}}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m)^{\mathrm{T}} = \\ &= \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \sum_{i=0,3} g_{ii}\gamma_i\ \eta_{0,p_1^{\mathrm{e}};0,p_1^{\mathrm{p}}}(t,z)\ \gamma_i^{\mathrm{T}}, \quad p_1^{\mathrm{e}} + p_1^{\mathrm{p}} = P_1.\end{aligned} \tag{72}$$

Here the superscript T denotes the transposition. With the help of the relation $\gamma_i^{\mathrm{T}} = -C^{-1}\gamma_i C$, with C being the charge conjugation matrix, $C^2 = 1$, and the anti-commutation relation $[\gamma_i, \gamma_5]_+ = 0$, $\gamma_5^2 = -1$, we may write for a new Bethe-Salpeter amplitude $\Theta(t,z)$, defined as

$$\Theta(t,z) = \ \eta_{0,p_1^{\mathrm{e}};0,p_1^{\mathrm{p}}}(t,z)\ C\gamma_5, \tag{73}$$

the equation

$$\begin{aligned}&(\mathrm{i}\overrightarrow{\hat{\partial}}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m)\Theta(t,z)(-\mathrm{i}\overleftarrow{\hat{\partial}}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m) = \\ &= \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \sum_{i=0,3} g_{ii}\gamma_i\Theta(t,z)\gamma_i, \quad p_1^{\mathrm{e}} + p_1^{\mathrm{p}} = P_1.\end{aligned} \tag{74}$$

The unknown function Θ here is a 4×4 matrix, which contains as a matter of fact only four independent components. In order to correspondingly reduce the number of equations in the set (74), one should note that the γ-matrix algebra in two-dimensional space-time should have only four basic elements. In accordance with this fact, only the matrices $\gamma_{0,3}$ are involved in (74). Together with the matrix $\gamma_0\gamma_3$ and the unit matrix I they form the basis, since $\gamma_{0,3} \cdot \gamma_0\gamma_3 = \gamma_{3,0}$, $\gamma_0^2 = -\gamma_3^2 = (\gamma_0\gamma_3)^2 = 1$,

$[\gamma_0, \gamma_3]_+ = [\gamma_{0,3}, \gamma_0\gamma_3]_+ = 0$. Using this algebra and the general representation for the solution

$$\Theta = aI + b\gamma_0 + c\gamma_3 + d\gamma_0\gamma_3, \tag{75}$$

one readily obtains a closed set of four first-order differential equations for the four functions a, b, c, d of t and z. The same set will be obtained, if one replaces in eq.(74) and (75) the 4×4 matrices by the Pauli matrices (28), subject to the same algebraic relations, according, for instance, to the rule: $\gamma_0 \Rightarrow \sigma_3$, $\gamma_3 \Rightarrow \mathrm{i}\sigma_2$, $\gamma_0\gamma_3 \Rightarrow \sigma_1$. Then equation (69) becomes a matrix equation

$$\begin{aligned}(\mathrm{i}\overrightarrow{\partial_t}\sigma_3 + \overrightarrow{\partial_z}\sigma_2 - \frac{P_0}{2}\sigma_3 + \frac{P_3}{2}\mathrm{i}\sigma_2 - m)\vartheta(t,z)(-\mathrm{i}\overleftarrow{\partial_t}\sigma_3 - \overleftarrow{\partial_z}\sigma_2 - \frac{P_0}{2}\sigma_3 + \frac{P_3}{2}\mathrm{i}\sigma_2 - m) = \\ = \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2 - t^2}{(eB)^2}}\,\{\sigma_3\vartheta(t,z)\sigma_3 + \sigma_2\vartheta(t,z)\sigma_2\}, \quad p_1^{\mathrm{e}} + p_1^{\mathrm{p}} = P_1.\end{aligned} \tag{76}$$

for a 2×2 matrix ϑ

$$\vartheta = aI + b\sigma_3 + \mathrm{i}c\sigma_2 + d\sigma_1. \tag{77}$$

Here I is the 2×2 unit matrix, and functions a, b, c, d are the same as in (75).

The following remark is in order. The derivation above relates to the system of two spinor fields, marked by the superscripts e and p, with opposite charges $\pm e$ and, generally, different masses. Although we kept the masses equal above, it is easy to restore their difference by setting $m = m^{\mathrm{e}}$ in the left differential operator and $m = m^{\mathrm{p}}$ in the right one starting with eq.(26) throughout. The corresponding Bethe-Salpeter amplitude η is the translationally invariant part (30) of the matrix element of the chronological product of the spinor field operators

$$\chi_{\beta\nu}(x^{\mathrm{e}}, x^{\mathrm{p}}) \;\; =< 0|T(\psi^{\mathrm{e}}_\beta(x^{\mathrm{e}})\psi^{\mathrm{p}}_\nu(x^{\mathrm{p}}))|P> \tag{78}$$

between the vacuum $< 0|$ and the bound state $|P>$. Once we restrict ourselves to the case where one of the particles, ψ^{e}, is an electron and the other, ψ^{p}, is a positron, we should take $\psi^{\mathrm{p}} = C\overline{\psi}^{\mathrm{e}}$ in (78) and keep the masses equal. Then the Bethe-Salpeter amplitude of two arbitrary opposite-charged fermions χ and the electron-positron Bethe-Salpeter amplitude $\varrho =< 0|T(\psi^{\mathrm{e}}_\beta(x^{\mathrm{e}})\psi^{\mathrm{e}}_\nu(x^{\mathrm{p}}))|P>$ are connected as $\chi = \varrho\, C^{\mathrm{T}} = -\varrho\, C$. It follows from (76) that the translationally-invariant part of the Ritus transform of ϱ obeys the same equation as (74), but with the sign in front of the hatted terms in the right-most Dirac operator reversed, as well as the common sign in the right-hand side. The subsequent γ_5-transformation in (73) is useful, since it gives the possibility to form the Laplace operator in the subsequent equations.

3.4. Including an external electric field

Let us generalize the two-dimensional Bethe-Salpeter equation obtained in the presence of a strong magnetic field by including an external electric field, parallel to it, that is not supposed to be strong, $E \ll B$. To this end we supplement the potential (29) in

equation (26) by two more nonzero components

$$A_0(x_0,x_3),A_3(x_0,x_3))\neq 0, \tag{79}$$

that carry the electric field - not necessarily constant - directed along the axis 3. We shall use the collective notations $A_\parallel=(A_0,A_3)$, $x_\parallel=(x_0,x_3)$, $\hat{\partial}_\parallel^{\rm e,p}=\partial_0^{\rm e,p}\gamma_0-\partial_3^{\rm e,p}\gamma_3$, $\hat{A}_\parallel=A_0\gamma_0-A_3\gamma_3$. We shall not exploit now a representation like (30), but deal directly with the Bethe-Salpeter amplitude $\chi(x^{\rm e},x^{\rm p})$ as a function of the electron and positron coordinates, and with its Fourier-Ritus transform $\chi_{h^{\rm e}h^{\rm p}}(x_\parallel^{\rm e},x_\parallel^{\rm p})$ connected with $\chi(x_\parallel^{\rm e},x_\parallel^{\rm p};x_\perp^{\rm e},x_\perp^{\rm p})$ in the same way as (32). In place of equation (31) one should write

$$\left[{\rm i}\hat{\partial}_\parallel^{\ \rm e}-e\hat{A}_\parallel(x_\parallel^{\rm e})-m+{\rm i}\hat{\partial}_\perp^{\rm e}-e\gamma_1A_1(x_2^{\rm e})\right]_{\alpha\beta}\left[{\rm i}\hat{\partial}_\parallel^{\ \rm p}+e\hat{A}_\parallel(x_\parallel^{\rm p})-m+{\rm i}\hat{\partial}_\perp^{\rm p}+e\gamma_1A_1(x_2^{\rm p})\right]_{\mu\nu}\cdot$$

$$\cdot\ \left[\chi(x_\parallel^{\rm e,p},x_\perp^{\rm e,p})\right]_{\beta\nu}=-{\rm i}8\pi\alpha D_{ij}(t,z,x_{1,2}^{\rm e}-x_{1,2}^{\rm p})\,[\gamma_i]_{\alpha\beta}\,[\gamma_j]_{\mu\nu}\,\left[\chi(x_\parallel^{\rm e,p},x_\perp^{\rm e,p})\right]_{\beta\nu}\ , \tag{80}$$

Thanks to the commutativity (43) the rest of the procedure of the previous Subsection remains essentially the same, and we come, in place of (69), to the following two-dimensional equation

$$\left[{\rm i}\hat{\partial}_\parallel^{\ \rm e}-e\hat{A}_\parallel(x_\parallel^{\rm e})-m\right]_{\alpha\beta}\left[{\rm i}\hat{\partial}_\parallel^{\ \rm p}+e\hat{A}_\parallel(x_\parallel^{\rm p})-m\right]_{\mu\nu}\ \left[\chi_{0,p_1^{\rm e};0,p_1^{\rm p}}(x_\parallel^{\rm e},x_\parallel^{\rm p})\right]_{\beta\nu}\ =$$

$$=\frac{2\alpha\pi^{-1}}{z^2+\frac{P_1^2}{(eB)^2}-t^2}\sum_{i=0,3}g_{ii}\,[\gamma_i]_{\alpha\beta}\,[\gamma_j]_{\mu\nu}\,\left[\chi_{0,p_1^{\rm e};0,p_1^{\rm p}}(x_\parallel^{\rm e},x_\parallel^{\rm p})\right]_{\beta\nu}\ , \tag{81}$$

for a positronium atom in a strong magnetic field placed in a moderate electric field, parallel to the magnetic one. In order to apply this equation to a system of two different oppositely charged particles interacting with each other through the photon exchange and placed into the combination of a strong magnetic and an electric field in the same direction, say a relativistic hydrogen atom, one should only distinguish the two masses in the first and second square brackets in the left-hand side.

4. Ultra-relativistic regime in a magnetic field

In the ultra-relativistic limit, where the positronium mass is completely compensated by the mass defect, $P_0=0$, for the positronium at rest along the direction of the magnetic field $P_3=0$, the most general relativistic-covariant form of the solution (75) is

$$\Theta=I\Phi+\hat{\partial}_\parallel\Phi_2+\gamma_0\gamma_3\Phi_3. \tag{82}$$

The point is that $\gamma_0\gamma_3$ is invariant under the Lorentz rotations in the plane (t,z). Substituting this into (74) with $P_0=P_3=0$ we get a separate equation for the singlet component of (82)

$$\left(-\Box_2+m^2\right)\Phi(t,z)=\frac{4\alpha\pi^{-1}\Phi(t,z)}{z^2+\frac{P_1^2}{(eB)^2}-t^2} \tag{83}$$

and the set of equations

$$\left(\Box_2+m^2\right)\Phi_3(t,z)=-\frac{4\alpha\pi^{-1}\Phi_3(t,z)}{z^2+\frac{P_1^2}{(eB)^2}-t^2},$$

$$(-\Box_2 + m^2)\partial_t \Phi_2 + 2m\mathrm{i}\partial_z \Phi_3 = 0,$$
$$(-\Box_2 + m^2)\partial_z \Phi_2 + 2m\mathrm{i}\partial_t \Phi_3 = 0 \tag{84}$$

for the other two components. Here $\Box_2 = -\partial^2/\partial t^2 + \partial^2/\partial z^2$ is the Laplace operator in two dimensions. Note the "tachyonic" sign in front of it in the first equation (84).

Let us differentiate the second equation in (84) over z and the third one over t and subtract the results from each other. In this way we get that $\Box_2\Phi_3 = 0$. This, however, contradicts the first equation in (84). Therefore, only $\Phi_3 = 0$ is possible. Then, the two second equations in (84) are satisfied, provided that $(-\Box_2 + m^2)\Phi_2 = 0$. We shall concentrate in equation (83) in what follows.

The longitudinal momentum along x_1, or the distance between the orbit centers along x_2, plays the role of the effective photon mass and a singular potential regularizator in equation (83). The lowest state corresponds to the zero value of the transverse total momentum $P_1 = 0$. In this case the equation (83) for the Ritus transform of the Bethe-Salpeter amplitude finally becomes

$$\left(-\Box_2 + m^2\right)\Phi(t,z) = \frac{4\alpha\Phi(t,z)}{\pi(z^2 - t^2)}. \tag{85}$$

4.1. Fall-down onto the center in the Bethe-Salpeter amplitude for high magnetic field. First hypercritical field.

We are going now to consider the consequences of the fall-down onto the center phenomenon present in equation (85), formally valid for an infinite magnetic field, and the alterations introduced by its finiteness.

In the most symmetrical case, when the wave function $\Phi(x) = \Phi(s)$ does not depend on the hyperbolic angle ϕ in the space-like region of the two-dimensional Minkowsky space, $t = s\sinh\phi$, $z = s\cosh\phi$, $s = \sqrt{z^2 - t^2}$ equation (85) becomes the Bessel differential equation

$$-\frac{\mathrm{d}^2\Phi}{\mathrm{d}s^2} - \frac{1}{s}\frac{\mathrm{d}\Phi}{\mathrm{d}s} + m^2\Phi = \frac{4\alpha}{\pi s^2}\Phi. \tag{86}$$

It follows from the derivation procedure in the previous Section 3 that this equation is valid within the interval

$$\frac{1}{\sqrt{eB}} \ll s_0 \le s \le \infty, \tag{87}$$

where the lower bound s_0 depends on the external magnetic field - it should be larger than the Larmour radius $L_B = (eB)^{-1/2}$ and tend to zero together with it, as the magnetic field tends to infinity. The stronger the field, the ampler the interval of validity, the closer to the origin $s = 0$ the interval of validity of this equation extends. If the magnetic field is not sufficiently strong, the lower bound s_0 falls beyond the region where the solution is mostly concentrated and the limiting form of the Bethe-Salpeter equation becomes noneffective, since it only relates to the asymptotic (large s) region, while the rest of the s-axis is served by more complicated initial Bethe-Salpeter equation, not

reducible to the two-dimensional form there. This is how the strength of the magnetic field participates - note, that the coefficients of eq.(86) do not contain it.

In treating the falling to the center below we shall be using s_0 as the lower edge of the normalization box (see the discussion in Section 2). For doing this it is necessary that s_0 be much smaller than the electron Compton length, the only dimensional parameter in equation (12). In this case the asymptotic regime of small distances is achieved and nothing in the region $s < s_0$ beyond the normalization volume - where the two-dimensional equations (69), (74), (83), (85) and hence (86) are not valid- may affect the problem, because this is left behind the event horizon.

In alternative to this, we might treat s_0 as the cut-off parameter. In this case we have had to extend equation (12) continuously to the region $0 \leq s \leq s_0$, simultaneously replacing the singularity s^{-2} in it by a model function of s, nonsingular in the origin, say, a constant s_0^{-2}. In this approach the results are dependent on the choice of the model function which is intended to substitute for the lack of a treatable equation in that region. Besides, the limit $s_0 \to 0$ does not exist. The latter fact implies that the approach should become invalid for sufficiently small s_0, i.e., large B. We, nevertheless, shall also test the consequences of this approach later in this section to make sure that in our special problem the result is not affected any essentially.

The crucial difference of (86) from equation (12), which relates to the case where the magnetic field is absent, is the coefficient 1 in front of the first-derivative term instead of 3 (this coefficient is $D-1$, where D is the dimension of the Minkowsky space in the radial part of the Laplacian $\Box_D = s^{-D+1}\partial/\partial s(s^{D-1}\partial/\partial s)$). The coefficient 4 in front of α in (85) and (86) instead of 8 in (12) is also of geometric origin: the relation

$$\sum_{i,j=0,3} g_{ij}\gamma_i\gamma_j = 2 \tag{88}$$

was used when we passed from (69) to (83), whereas relation (10) was exploited to pass from (2) to (12). Solutions of (86) behave near the singular point $s = 0$ like s^σ, where

$$\sigma = \pm 2\sqrt{-\frac{\alpha}{\pi}}. \tag{89}$$

The fall-down onto the center [1] occurs, if $\alpha > \alpha_{\rm cr} = 0$, *i.e.,* unlike (18), for arbitrary small attraction, the genuine value $\alpha = 1/137$ included. With the substitution $\Phi(s) = \Psi(s)/\sqrt{s}$ equation (86) acquires the standard form of a Schrödinger equation

$$-\frac{\mathrm{d}^2\Psi(s)}{\mathrm{d}s^2} + \frac{-4\frac{\alpha}{\pi} - \frac{1}{4}}{s^2}\Psi(s) + m^2\Psi(s) = 0, \qquad s_0 \leq s \leq \infty, \quad s_0 \gg (eB)^{-1/2} \tag{90}$$

Treating the applicability boundary s_0 of this equation as the lower edge of the normalization box, as discussed above, $s_0 \ll m^{-1}$, we impose the standing wave boundary condition (19) on the solution of (90)

$$\Psi(s) = \sqrt{s}\, K_\nu(ms), \quad \nu = \mathrm{i}2\sqrt{\frac{\alpha}{\pi}} \tag{91}$$

that decreases at infinity.

Starting with a certain small value of the argument ms, the McDonald function with imaginary index (91) oscillates, as $s \to 0$, passing the zero value infinitely many times. Therefore, if s_0 is sufficiently small the standing wave boundary condition (19) can be definitely satisfied. Keeping to the genuine value of the coupling constant $\alpha = 1/137$ ($\nu = 0.096$ i) one may ask: what is the largest possible value $s_0^{\max}$ of s_0, for which the boundary problem (90), (19) can be solved? By demanding, in accord with the validity condition (87) of equation (86), (90), that the value of $s_0^{\max}$ should exceed the Larmour radius

$$s_0^{\max} \gg (eB)^{-1/2} \quad \text{or} \quad B \gg \frac{1}{e\,(s_0^{\max})^2} \tag{92}$$

one establishes, how large the magnetic field should be in order that the boundary problem might have a solution, in other words, that the point $P_0 = \mathbf{P} = 0$ might belong to the spectrum of bound states of the Bethe-Salpeter equation in its initial form (26).

One can use the asymptotic form of the McDonald function near zero to see that the boundary condition (19) is satisfied provided that

$$\left(\frac{ms_0}{2}\right)^{2\nu} = \frac{\Gamma(1+\nu)}{\Gamma^*(1-\nu)} \tag{93}$$

or

$$\nu \ln \frac{ms_0}{2} = \mathrm{i} \arg \Gamma(\nu+1) - \mathrm{i}\pi n, \qquad n = 0, \pm 1, \pm 2, ... \tag{94}$$

Once $|\nu|$ is small we may exploit the approximation (23) for the Γ-function to get

$$\ln\left(\frac{ms_0}{2}\right) = -\frac{n}{2}\sqrt{\frac{\pi^3}{\alpha}} - C_{\mathrm{E}}, \quad n = 1, 2, ... \tag{95}$$

We have expelled the non-positive integers n from here for the same reasons as in Section 2 (see eq.(24)). The maximum value for s_0 is provided by $n = 1$. The Euler constant C_{E}=0.577 contribution is small as compared to $(1/2)\sqrt{\pi^3/\alpha} = 32.588$. We finally get

$$\ln\left(\frac{ms_0^{\max}}{2}\right) = -\frac{1}{2}\sqrt{\frac{\pi^3}{\alpha}} - C_{\mathrm{E}}$$

or

$$s_0^{\max} = \frac{2}{m}\exp\left\{-\frac{1}{2}\sqrt{\frac{\pi^3}{\alpha}} - C_{\mathrm{E}}\right\} \simeq \frac{2}{m}\exp\{-33\} \simeq 10^{-14}\frac{1}{m}. \tag{96}$$

This is fourteen orders of magnitude smaller than the Compton length $m^{-1} = 3.9 \cdot 10^{-11}$cm and makes about 10^{-25}cm. Now, in accord with (92), if the magnetic field exceeds the first hypercritical value of

$$B_{\mathrm{hpcr}}^{(1)} = \frac{m^2}{4e}\exp\left\{\frac{\pi^{3/2}}{\sqrt{\alpha}} + 2C_{\mathrm{E}}\right\} \simeq 1.6 \times 10^{28}\ B_0, \tag{97}$$

the positronium ground state with the center-of-mass 4-momentum equal to zero appears. Here $B_0 = m^2/e = 2.17 \times 10^{21}\mathrm{cm}^{-2}$ is the Schwinger critical field, or $B_0 = m^2c^3/e\hbar = 1.22 \times 10^{13}$ Heaviside-Lorentz units, or $B_0 = 4.4 \times 10^{13}$Gauss ($\alpha = e^2/4\pi\hbar c$, $e = 4.8\cdot 10^{-10}\sqrt{4\pi}$ CGSE). Excited positronium states may also reach the spectral point $P_\mu = 0$, but this occurs for magnetic fields, tens orders of magnitude larger

than (97) - to be found in the same way from (95) with $n = 2, 3...$ The ultra-relativistic state $P_\mu = 0$ has the internal structure of a confined state, i.e. the one whose wave function behaves as a standing wave combination of free particles near the lower edge of the normalization box and decreases as $\exp(-ms)$ at large distances. The effective "Bohr radius", i.e. the value of s that provides the maximum to the wave function (91) makes $s_{\max} = 0.17m^{-1}$ (this fact is established by numerical analysis). This is certainly much less than the standard Bohr radius $(e^2m)^{-1}$. Taken at the level of 1/2 of its maximum value, the wave-function is concentrated within the limits $0.006\ m^{-1} < s < 1.1\ m^{-1}$. But the effective region occupied by the confined state is still much closer to $s = 0$. The point is that the probability density of the confined state is the wave function squared *weighted with the measure* $s^{-2}\mathrm{d}s$ *singular in the origin [13], [5]* and is hence concentrated near the edge of the normalization box $s_0 = 10^{-25}$cm, and not in the vicinity of the maximum of the wave function. The electric fields at such distances are about 10^{43} volt/cm. Certainly, there is no evidence that the standard quantum theory should be valid under such conditions. This remark gives the freedom of applying the theory in Refs. [13], [5].

It is interesting to compare the value (97) with the analogous value, obtained earlier by the present authors (see p.393 of Ref.[9]) by extrapolating the nonrelativistic result concerning the positronium binding energy in a magnetic field to extreme relativistic region:

$$B_{\rm hpcr}|_{\rm NONRELATIVISTIC} = \frac{\alpha^2 m^2}{e} \exp\left\{\frac{2\sqrt{2}}{\alpha}\right\} = B_0 \cdot 10^{164}. \tag{98}$$

Such is the magnetic field that makes the binding energy of the lowest energy state equal to (-2m). (This is worth comparing with the magnetic field, estimated [27] as $\alpha^2\exp(2/\alpha)B_0$, that makes the mass defect of the nonrelativistic hydrogen atom comparable with the electron rest mass. A more exact nonrelativistic value for this quantity may be found using the asymptotic consideration in [28]). We see that the relativistically enhanced attraction has resulted in a drastically lower value of the hypercritical magnetic field. Note the difference in the character of the essential nonanalyticity with respect to the coupling constant: it is $\exp(\pi\sqrt{\pi}/\sqrt{\alpha})$ in (97) and $\exp(2\sqrt{2}/\alpha)$ in (98). Another effect of relativistic enhancement is that within the semi-relativistic treatment of the Bethe-Salpeter equation [9], as well as within the one using the Schrödinger equation [6], only the lowest level could acquire unlimited negative energy with the growth of the magnetic field, whereas according to (95) in our fully relativistic treatment all excited levels with $n > 1$ are subjected to the falling to the center and can reach in turn the point $P_\| = 0$.

Let us see now, how the result (97) is altered if the cut-off procedure of Ref.[1] is used. Consider equation (90) in the domain $s_0 < s < \infty$, but replace it with another equation

$$-\frac{\mathrm{d}^2\Psi_0(s)}{\mathrm{d}s^2} - \frac{\frac{4\alpha}{\pi} + \frac{1}{4}}{s_0^2}\Psi_0(s) + m^2\Psi_0(s) = 0 \tag{99}$$

in the domain $0 < s < s_0$. The singular potential is replaced by a constant near the origin in (99). Demand, in place of (19), that $\Psi_0(0) = 0$, $(\Psi_0'(s_0)/\Psi_0(s_0)) = (\Psi'(s_0)/\Psi(s_0))$. Then the result (97) will be modified by the factor

$$\exp\left\{-\frac{2}{\sqrt{\frac{4\alpha}{\pi}+\frac{1}{4}}\,\cot\left(\frac{4\alpha}{\pi}+\frac{1}{4}\right)-\frac{1}{2}}\right\}, \tag{100}$$

which may be taken at the value $\alpha = 0$. Thus, the result (97) is only modified by a factor of $\exp(-4/3) \simeq 0.25$. Generally, the estimate of the limiting magnetic field (97) is practically nonsensitive to the way of cut-off, in other words to any solution of the initial equation inside the region $0 < s < s_0$, where the magnetic field does not dominate over the mutual attraction force between the electron and positron. This fact takes place, because the term $(\pi^{3/2}\sqrt{\alpha}) \simeq 65$, singular in α, is prevailing in (97), the details of the behavior of the wave function close to the origin $s = 0$ being not essential against its background. Although numerically the resulting value of the crucial magnetic field is affected but very little, we must bear in mind that the whole cut-off approach is not adequate, as argued in Section 2, and is burdened by the blind extension of the standard quantum mechanics to the situation, where electron and positron are brought together closer than $10^{-14}m^{-1}$! The ultra-relativistic state $P_\mu = 0$ arising within this approach is an ordinary bound state, not the confined state described above.

4.2. Radiative corrections

Mass radiative corrections should be taken into account by inserting the mass operator into the Dirac differential operators in the l.-h. sides of the Bethe-Salpeter equation (26) or (69). We shall estimate now, whether this may affect the above conclusions concerning the positronium mass compensation by the binding energy. It was believed that the radiative corrections to the electron mass are able themselves to annihilate the electron mass due to the interaction of anomalous magnetic moment with the external magnetic field. However, this moment is not constant, but becomes negative for sufficiently strong magnetic field [29]. So we are left with the primary result that in the strong magnetic field the mass of an electron in Landau ground state grows with the field B as [30]

$$\widetilde{m} = m\left(1 + \frac{\alpha}{4\pi}\ln^2\frac{B}{B_0}\right). \tag{101}$$

For $B \simeq B_{\rm hpcr}^{(1)}$ the corrected mass makes $\widetilde{m} = 3.45m$. This implies that the mass annihilation due to the falling to the center requires a field somewhat larger than (97). To determine its value, substitute $\widetilde{m}$ (101) for m and $L_B = (eB)^{-1/2}$ for s_0 into equation (95) with $n = 1$. The resulting equation for the first hypercritical magnetic field, modified by the mass radiative corrections, $B_{\rm corr}$

$$\left(1 + \frac{\alpha}{4\pi}\ln^2\frac{B_{\rm corr}}{B_0}\right) = 4\frac{B_{\rm corr}}{B_0}\exp\left\{\frac{1}{2}\sqrt{\frac{\pi^3}{\alpha}} + C_{\rm E}\right\} \tag{102}$$

has the numerical solution: $B_{\rm corr} = 13\ B_{\rm hpcr}^{(1)}$.

We state that this correction, increasing the first hypercritical value $B^{(1)}_{\rm hpcr}$ by a little more than one order of magnitude, is not essential bearing in mind the huge values (97) of the latter.

4.3. Second hypercritical field.

The same as in Section 2, we may attribute the O(1.1)- symmetrical solution (91), which is a spinorial singlet, to the vacuum. Its 2-momentum quantum numbers $P_\|$ are zero, which means that the vacuum state is constant with respect to the center-of-mass position $x^{\rm e}+x^{\rm p}$ of its constituents. On the contrary, with respect to the space-like interval s between the vacuum constituents the wave function decreases if these are taken apart, as stated above in Subsection 4.1; this means that these constituents are strongly localized. (The vacuum constituents may be thought of as delocalized in the "internal coordinate space" obtained by mapping the singular point of coincidence $s=0$ to the negative infinity - see Refs. [13], [5] for a detailed explanation of associated matters).

In this subsection we discuss in a qualitative way the situation that may take place when the magnetic field exceeds the first hypercritical value (97). The eigenvalues of the Bethe-Salpeter equation (69) for the total 2-momentum components $P_{0,3}$ of the $\rm e^+$-$\rm e^-$ system are now expected to shift into the space-like region, whereas for $B<B^{(1)}_{\rm hpcr}$ the center-of-mass 2-momentum of the then real pair was, naturally, time-like.

With $P_{0,3}\neq 0$ equation (69) becomes more complicated as compared to the case $P_{0,3}=0$ considered above in this section. So, decomposition (82) is no longer sufficient, but should be supplemented by an extra term $\hat P_\|\Phi_4$. The resulting set of equations for Φ's does not split now, unlike the set (83), (84) did. Nevertheless, at least for far space-like $P_\|$, $P_\|^2\ll -4m^2$, the situation can be modelled by the same equation as (90), but with the large negative quantity $m^2+P_\|^2/4$ substituted for m^2. Then the McDonald function (91) is replaced by the Hankel function containing two oscillating exponentials for large space-like intervals s

$$\exp\left\{\pm {\rm i}s\sqrt{\left|m^2+\frac{P_\|^2}{4}\right|}\right\}\exp\left\{{\rm i}P_\|\frac{x_{\rm e}+x_{\rm p}}{2}\right\}. \tag{103}$$

and two oscillating exponentials

$$s^{\pm 2{\rm i}\sqrt{\frac{\alpha}{\pi}}} \tag{104}$$

for small ones. If one passes to the Lorentz frame, where $P_0=0, P_3\neq 0$, and sets the time arguments in the two-time Bethe-Salpeter amplitude equal to one another: $x_0^{\rm e}=x_0^{\rm p}$, one finds that the solution oscillates along the magnetic field with respect to the relative coordinate $x_3^{\rm e}-x_3^{\rm p}$ (mutually free particles) and with respect to the c.m. coordinate $x_3^{\rm e}+x_3^{\rm p}$ (vacuum lattice).

We are now in the kinematical domain called sector IV, or deconfinement sector in Refs.[13], [5]. In this sector the constituents are free at large intervals and near the singular point $s=0$. The wave incoming from infinity is partially reflected, but

partially penetrates to the singular point, the probability of creation of the delocalized (free) states is determined by the barrier transmission coefficient [5]. Such states may exist if one succeeds to satisfy self-adjoint boundary conditions . These are, for instance, periodic conditions, to be imposed on the lower and upper boundaries of the normalization volume, in stead of the standing wave condition (19), appropriate in sector III. The corresponding eigenfunctions are studied in [13]. The possibility to obey them is provided again by the falling to the center.

Now, the delocalized states in two-dimensional Minkowsky space correspond to electron and positron that circle along Larmour orbits with very small radii in the plane orthogonal to the magnetic field and simultaneously perform, when the interval between them is large, a free motion along the magnetic field. They possess magnetic moments and seem to be capable of screening the magnetic field. This provides the mechanism that prevents the classical magnetic field from being larger than the hypothetical value, second hypercritical field, for which the delocalization first appears.

The tachyonic character of the vacuum state, i.e. the space-likeness of its 2-momentum quantum number, does not make a difficulty. The presence of this quantum number implies the break down of the invariance under the Lorentz transformation along the magnetic field due to the appearance of the lattice in the frame $P_0 = 0$ or of a superluminal wave in arbitrary frame. This is not a contradiction, since such a wave appears in response to a simultaneous increase of the constant magnetic field in the whole space, which is already a noncausal procedure.

5. Conclusion

In Section 3 we derived the fully relativistic two-dimensional form that the differential Bethe-Salpeter equation for the electron-positron system takes in the limit of infinite constant and homogeneous magnetic field imposed on the system. We studied the falling to the center phenomenon inherent in this equation basing on exactly relativistic treatment of the relative motion of the electron and positron. Thanks to this phenomenon, at a certain finite value of the magnetic field (97) called here the *first hypercritical value* the positronium level deepens so much that the rest energy of the system is completely compensated for by the mass defect. The most symmetrical solution of the Bethe-Salpeter equation corresponding to the center-of-mass momentum equal to zero may be attributed to the vacuum. In Subsection 4.1 we described the vacuum restructuring that takes place after the magnetic field exceeds the first hypercritical value in terms of formation of localized states of the pair, which are either "confined" or tightly bound - depending on whether the theory of the falling to the center in Refs. [13], [5] is appealed to or not. We estimate in Subsection 4.2 the modification of the first hypercritical value of the magnetic field that may be introduced by the mass corrections to the Dirac field propagator in the strong magnetic field. In Subsection 4.3 we discuss the *second hypercritical value* of the magnetic field where a lattice appears in the vacuum and the latter becomes unstable under the delocalization

of the states of the pair, the delocalized charged particles on the Larmour orbits being capable of screening the external field and thus setting a limit to its growth.

The above limiting values are obtained within pure quantum electrodynamics. Up to now, it was accepted that the vacuum of this theory is stable with any magnetic field, contrary to electric field and contrary to non-Abelian gauge field theories like QCD. In spite of the huge values, expected to be present, perhaps, only in superconducting cosmic strings [31], the values obtained may be important as setting the limits of applicability of QED.

As being due to the special, non-perturbational mechanism described above, the hypercritical field is determined by the inverse square root of the fine structure constant elevated to the exponent. This makes it hundred or so orders of magnitude smaller than other known typical values [32] of the magnetic field that may be expected to lead us beyond the scope of coverage of QED owing to the lack of asymptotic freedom. For instance [33], the photon becomes a tachyon in the magnetic field of the order of $B_0 \exp(3\pi/\alpha)$.

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